# Burdett-Mortensen Model of On-the-Job Search with Two Sectors<sup>\*</sup>

Florian Hoffmann<sup>a</sup>

(florian.hoffmann@ubc.ca) <sup>a</sup>Vancouver School of Economics 997 – 1873 East Mall, Vancouver, BC Canada, V6T 1Z1 Shouyong Shi<sup>b</sup> (sus67@psu.edu) <sup>b</sup>Department of Economics Pennsylvania State University University Park, Pennsylvania USA, 16802

This version: October 2015

#### Abstract

The focus of this paper is on the steady state of a two-sector economy with undirected search where employed and unemployed workers can search for jobs, both within a sector and between the sectors. As in the one-sector model, on-the-job search generates wage dispersion among homogeneous workers. The analysis of the two-sector model uncovers a property called constant tension that is responsible for analytical tractability. We characterize the steady state in all cases with constant tension. When time discounting vanishes, constant tension yields the endogenous separation rate in each sector as a linear function of the present value for a worker. The one-sector economy automatically satisfies constant tension, in which case the linear separation rate implies that equilibrium offers of the worker value are uniformly distributed. Constant tension also has strong predictions for worker transitions and value/wage dispersion, both within a sector and between the two sectors. When constant tension does not hold, we compute the steady state numerically and illustrate its properties.

Keywords: Search on the job; Wage dispersion; Constant tension.

JEL codes: E24, J60

<sup>\*</sup> Corresponding author: Shouyong Shi. We thank two anonymous referees for helpful comments. The authors started the work on this paper when both were at University of Toronto. We are grateful to the Social Sciences and Humanities Research Council of Canada for financial support. In addition, Shi gratefully acknowledges financial support from the Canada Research Chair, the Bank of Canada Fellowship, and Pennsylvania State University. The view expressed here is the authors' own and does not reflect the view of the Bank of Canada.

# 1. Introduction

In a seminal paper, Burdett and Mortensen (1998, BM henceforth) explore on-the-job search as a mechanism for generating job-to-job flows of workers and wage dispersion simultaneously. Their analysis reveals a deep insight about the interaction between competition and search frictions. On-the-job search generates the possibility that a worker may leave a firm for other firms. Facing this possibility, a firm makes the optimal tradeoff between recruiting and retaining a worker, on the one hand, and ex post profit on the other hand. As a result of this tradeoff, equilibrium wage rates must be distributed continuously in an interval, even though all workers and all firms are homogeneous.<sup>1</sup>

The BM model has found wide applications in labor and macroeconomics. It has been used to explain residual wage inequality, i.e., the wage differential among workers that seems difficult to be explained by worker and firm characteristics (van den Berg and Ridder, 1998; Bontemps et al., 2000; Mortensen, 2005). Also, by linking wage dispersion tightly to on-the-job search, the BM model offers a natural framework for jointly studying job mobility, wage dynamics, and the sorting pattern between firms and workers (Postel-Vinay and Robin, 2002; Jolivet et al., 2006; Lise and Robin, 2013; Moscarini and Postel-Vinay, 2013). Moreover, the BM model is extended to explain the interaction between job mobility and wage-tenure contracts (Burdett and Coles, 2003).

In this paper, we analyze a BM model with two sectors that compete for the same pool of workers. There are two motivations for this study. One is theoretical and is to understand why the steady state of the BM model is analytically tractable despite that a non-degenerate wage distribution arises endogenously from on-the-job search. To explore the model, we reformulate an offer as the worker value, i.e., the present value delivered to a worker by a wage rate conditional on the worker's optimal separation in the future. This follows recent developments in the search literature (e.g., Burdett and Coles, 2003; Shi, 2009; Menzio and Shi, 2011). With homogeneous workers, dispersion in worker values

<sup>&</sup>lt;sup>1</sup>In a related model of the product market, Burdett and Judd (1983) generate a non-degenerate distribution of prices by assuming that some buyers can receive two price quotes before selecting one. The outside offer that an employed worker receives in the BM model acts like the second price quote.

represents true residual inequality, whereas dispersion in wage rates may not necessarily do so. With two sectors, in particular, a higher wage is not necessarily a better offer to a worker, depending on which sector the offer comes from, but a higher value is always a better offer. Thus, reformulating an offer as the worker value simplifies a worker's transition decision: a worker accepts higher values no matter which sector they come from. With this reformulation, we discover a feature called constant tension that is responsible for analytical tractability and, importantly, this role of constant tension becomes apparent only if one considers a two-sector model (see the description later). The one-sector BM model has constant tension automatically.

Our second motivation is empirical and is to understand how on-the-job search affects equilibrium dispersion and worker mobility between sectors when firms in different sectors compete for workers in the same labor market.<sup>2</sup> As documented by Lee and Wolpin (2006), there are direct transitions of workers between sectors. In a two-sector BM model, these between-sector transitions generate a link between the entire wage distributions in the sectors. This link is consistent with the empirical analysis in Hoffmann and Shi (2011), who show that rising monthly transition rates of employed workers from the non-service sector to the service sector over the last four decades were accompanied by a faster increase in residual wage inequality in the service sector. A two-sector BM model is a natural candidate for organizing these facts. Before enriching the model with worker and firm heterogeneity, it is useful to examine its analytical properties with homogeneous workers and firms first, which we do in this paper.

In our model, as in BM, both employed workers and unemployed workers can search, each firm offers and commits to a constant wage rate over the worker's employment in the firm, and the offer arrival rate from a sector to a worker is exogenous. In contrast to BM, there are two sectors (A and B). A firm can create a vacancy in either sector and a worker can receive an offer from either sector. The two sectors can differ in the value of output

<sup>&</sup>lt;sup>2</sup>Beaudry, et al. (2012) use a multi-sector search model to show that even without direct employment transitions between sectors or on-the-job search, there is an equilibrium force that affects the industrial composition and inter-sectoral wage differentials. In an extensive empirical analysis they show this force to be quantitatively important.

produced in a match, in the offer arrival rates, and in the rate of exogenous separation into unemployment. Moreover, the rate at which an employed worker receives an offer from the incumbent sector may differ from the rate of an offer from the other sector. We assume that a job in sector A produces a higher value than a job in sector B, refer to sector A as the high-productivity sector and sector B as the low-productivity sector.

In each sector, there is an interval of offers (of the worker value) that are optimal for a firm. A high offer increases the success in recruiting and reduces the endogenous quit rate of the worker in the future, at the cost of a lower profit flow of the job. All optimal offers maximize expected profit of a vacancy in a sector. In addition, because a firm can locate the vacancy in either sector, expected profit of a vacancy is equalized between the two sectors in the equilibrium. Competition among firms endogenously determines the relative size between the two sectors and the distribution of workers over values within each sector in the steady state. Because workers can receive offers from either sector, the hiring rate of a vacancy depends on the employed distributions of workers over values in both sectors. Similarly, the separation rate of a worker, which includes both exogenous separation and endogenous quits, depends on the offer distributions in both sectors.

A notable property is that the product of the hiring rate of a vacancy and the separation rate of a worker is constant over worker values in the support of the distribution in a sector. We refer to this property as *constant tension* in the sector and show that it makes the steady state tractable. Equivalently, constant tension requires the marginal intensities of hiring and separation to be equal to each other at every worker value in the support of the distribution in a sector. With constant tension, the hiring rate of a vacancy at each offer in a sector is tied to the separation rate at that offer in that sector. As all endogenous elements of expected profit of a vacancy are related exclusively to the separation rate, the condition of equal expected profit of a vacancy solves the separation rate as a function of the offer. Moreover, in the limit where time discounting, r, goes to zero, the solution of the separation rate in each sector is a linear function of the worker value. We examine all (three) cases of the steady state with constant tension. The first case is the one-sector BM model. An isolated sector automatically has constant tension, because the marginal intensities of hiring and separation must be equal to each other in order to maintain the steady state in a one-sector economy. Constant tension is the source of analytical tractability of the one-sector BM model. Not only can the steady state be solved analytically, but also the equilibrium offer distribution has a strikingly simple form. The linear solution of the separation rate immediately implies that offer values are uniformly distributed in the limit  $r \to 0$ . In contrast, wage rate offers are distributed according to a square-root function, as is well known from BM.

The second case is one in which workers employed in one sector cannot search in the other sector while unemployed workers can search in both sectors. Despite the restrictive nature, this case encompasses all multi-sectoral search models without on-the-job search (e.g., Beaudry, et al., 2012) as a special case and reveals equilibrium forces that are absent in these studies. Specifically, employed workers' search within their incumbent sector affects expected profit of a vacancy, thereby affecting the distributions of wage rates/values in both sectors and all workers' transitions. In the limit  $r \rightarrow 0$ , value offers are uniformly distributed in each sector. The lowest value offered in the two sectors is the same, which implies that the two sectors offer different ranges of values/wages. We find the necessary and sufficient conditions under which the range of values/wages is wider in the high-productivity sector than in the low-productivity sector. Moreover, we show that an increase in the unemployment benefit affects the two sectors differently in the steady state, including the effect on the transition rates of workers.

The third case is one in which the distributions of worker values in the two sectors do not overlap in the interior of their supports. In this case, the highest value in the low-productivity sector B is equal to the lowest value in the high-productivity sector A. Between-sector search by employed workers restricts the distributions of workers in the two sectors. Even if a firm in sector A can reduce the offer below the highest value offered in sector B, it is not optimal to do so because the lower offer reduces the firm's hiring rate from the other sector and increases future separation of a worker from the firm into the other sector. Not surprisingly, this non-overlapping steady state exists with two active sectors if and only if the productivity differential between the two sectors is neither too large nor too small, where the bounds on this differential depend on the parameters of search and exogenous separation. Although worker values do not overlap between the two sectors, wage rates in the two sectors can overlap. We analyze how an increase in the unemployment benefit affects the ranges of wage rates and values offered in each sector.

When the sectors do not have constant tension, we are not able to solve the steady state analytically. In this case, we compute the model numerically to illustrate the properties of the steady state.

Since this paper builds on BM, it is related to the large literature that has extended BM in a one-sector environment; see some papers cited earlier. The literature on BM with more than one sector is relatively small. Burdett (2012) seems to be the first one to extend BM to have two sectors: a private sector and a public sector. He assumes that the wage rate in the public sector is exogenously fixed at one level and examines how this public-sector wage rate affects wage dispersion and worker mobility in the private sector. Bradley, et al. (2014) extend Burdett's model to allow the distribution of wage rates in the public sector to be non-degenerate and estimate the structural parameters of the model. They keep the assumption that the wage distribution in the public sector is exogenous. Meghir, et al. (2015) formulate a model with a formal and an informal sector and estimate the structural parameters from Brazilian data. Using the method in Bontemps, et al. (2000), they estimate the employed wage distributions from the data, take these employed distributions as given and then solve the endogenous offer distributions under some parametric restrictions.

In contrast to these papers, we endogenize both the offer and the employed wage distributions in both sectors. The joint determination of these four equilibrium distributions is the main challenge to the analysis of a two-sector BM model, but it has two major advantages over the approach in Meghir, et al. (2015) or Bradley, et al. (2014) who estimate two of these four objects from the data and taking them as given. First, it is this joint determination that captures the strategic interactions between firms posting wages in either sector and competing for the same workers that generate a deep inter-dependence between the sector specific wage structures and various worker flows. As we show, the types of equilibria that can arise depend crucially on the assumptions on the parameters one is willing to make. Such results remain hidden if one does not solve the four equations jointly. Second, by determining all of the distributions jointly rather than taking some of them as given, our approach is suitable for a wide set of counterfactual policy analyses.

Another distinction is our analytical focus. This focus leads to the finding that constant tension is the source of analytical tractability of the steady state in the BM model and to the simple solutions of the separation rate and the offer distribution in each sector. We hope that these results greatly expand the knowledge of the BM model and will have independent use beyond this particular model.

## 2. The Model

#### 2.1. The Model Environment

Time is continuous. The economy is populated by a unit mass of workers and a unit mass of firms.<sup>3</sup> Workers and firms are risk neutral and discount future at the rate r. There are two sectors, indexed by  $i \in \{A, B\}$ . For each sector i, let -i denote the other sector. A worker can work, and a firm can create a vacancy, in either sector. As a result, the distributions of workers and firms between the two sectors are endogenous in the equilibrium. Let  $m_i$  be the mass of firms in sector i,  $n_i$  the mass of workers employed in sector i, and u the mass of unemployed workers. Then,  $m_A + m_B = 1$  and  $n_A + n_B + u = 1$ . We assume that a firm can post only one vacancy at a time and treats its filled jobs independently.<sup>4</sup> All workers

<sup>&</sup>lt;sup>3</sup>For simplicity and comparison with BM, we fix the total measure of firms in the economy and focus on the endogenous distribution of firms between the two sectors. It is not difficult to endogenize this measure by allowing for free entry of firms.

<sup>&</sup>lt;sup>4</sup>This assumption contrasts with BM who emphasize the tradeoff between the wage offer and firm size in the limit  $r \to 0$ . Although their analysis in this limit can be extended to two sectors, it becomes complicated when r > 0. In particular, if a firm in one sector has no presence in the other sector but deviates to fill a vacancy in the other sector, the firm will start in the other sector with a size different from other firms in that sector. It is difficult to deal with this phenomenon out of the steady state. By assuming that all jobs in a firm are treated independently, we eliminate this difficulty. Despite this different

can search for jobs. An unemployed worker enjoys a flow of income, b.

The two sectors differ from each other in three possible ways. First, the value of output produced in a match is sector specific and denoted  $y_i$  for sector *i*. We refer to  $y_i$  as sector *i*'s productivity (see the discussion later) and assume  $y_A > y_B > b$ . Sector *A* is the *high-productivity sector* and sector *B* the *low-productivity sector*. Second, in addition to endogenous separation induced by workers' search on the job, a worker-job pair in sector *i* exogenously separates at the rate  $\delta_i$ . Third, the two sectors may differ in search rates. An unemployed worker receives an offer from sector *i* at the rate  $\alpha_i$ . A worker employed in sector *i* receives an offer from sector *i* at the rate  $\alpha_i$ . A worker employed in sector *i* receives an offer from sector *i* at the rate  $\lambda_i$  and from the other sector at the rate  $\sigma_{-i}$ . The rates of search and exogenous separation are listed in Table 1. Let us refer to  $\lambda_A$  and  $\lambda_B$  as within-sector search rates, and  $\sigma_A$  and  $\sigma_B$  as between-sector search rates. The rates  $\sigma_A$  and  $\sigma_B$  can be positive or zero, but  $\alpha_i$ ,  $\delta_i$  and  $\lambda_i$  are strictly positive and finite for both i = A, B. For the general analysis, we do not restrict the relative magnitude of between-sector search rates to within-sector search rates.

rasie in ratio of search and energeneas separation							
workers' $\setminus$ rates	separation into	arrival rate of an offer from					
status	unemployment						
		Sector A firms	Sector B firms				
Unemployed		$\alpha_A$	$\alpha_B$				
Employed in sector A	$\delta_A$	$\lambda_A$	$\sigma_B$				
Employed in sector B	$\delta_B$	$\sigma_A$	$\lambda_B$				

Table 1. Rates of search and exogenous separation

Productivity in this model is specific to a sector, but not to a worker, a firm, or a match. This assumption is not meant to dismiss the empirical relevance of heterogeneity originated in workers, firms or matches. Rather, the assumption enables us to focus on how the introduction of sectoral productivity into BM can affect wage/value dispersion and worker flows both within a sector and between the two sectors. If productivity depends on workers, firms, or matches, wage/value dispersion can arise even without the mechanism emphasized by BM. As an empirical issue, different sectors may use different production

assumption, our analysis preserves the spirit of BM's analysis by emphasizing the tradeoff between the wage offer and the probability of filling a vacancy.

technologies. If so, then different sectors can have different levels of productivity even after controlling for observable differences in workers, firms and matches. Moreover, different sectors may produce different goods or services that differ in prices. Although y is referred to as productivity, it is the value of output produced in a match. Hence, the difference in product prices between the two sectors can be an important cause of the difference in y.

The assumption of fixed  $(\alpha, \lambda, \sigma)$  implies that the measure of meetings between workers and sector *i* firms is  $(\alpha_i u + \lambda_i n_i + \sigma_i n_{-i})$ . This meeting function has constant returns to scale in  $(u, n_i, n_{-i})$  but is independent of  $m_i$ , the measure of vacancies in sector *i*. The meeting rate of a vacancy in sector i with an unemployed worker, for example, is  $\alpha_i u/m_i$ , which has elasticity -1 with respect to  $m_i$ . It is undoubtedly more realistic to specify a general meeting function so as to make the arrival rate of an offer from sector i to a worker increase in  $m_i$ , in which case the meeting rate of a vacancy in sector i is less elastic with respect to  $m_i$ . We keep the simple specification for a number of reasons. First, the general specification introduces non-linearity in  $(n_i, n_{-i}, m_i)$  that contributes little to the understanding of when the steady state is tractable. As will become clear later, the between-sector dependence of the hiring and separation rates on the distributions of values, not the dependence on the measures of firms and workers, is the key obstacle to tractability. Second, the simple specification emphasizes competition among firms with a high elasticity of the meeting rate of a vacancy. This emphasis seems appropriate in this model because firms, not workers, are assumed to be able to choose between the two sectors before matching occurs. The specification is also convenient for incorporating free entry of vacancies in the economy, although this extension is omitted here to save space. Third, the specification ensures comparability with BM. In our model, it is straightforward to shut down employed workers' search between the sectors by setting  $\sigma_A = \sigma_B = 0$ .

As in BM, we focus on the steady state and maintain the following assumptions. First, search is undirected in the sense that a worker does not choose which offer to apply for; instead, an offer arrives at an exogenous rate described above. Second, firms do not respond to a worker's outside offer. Instead, a worker who receives an outside offer decides either to take the offer and quit the current job, or reject the offer and stay at the current job. Third, an offer is a wage rate that is constant during the worker's employment in the firm.<sup>5</sup>

A wage rate w yields an expected lifetime utility or value, V, given the worker's optimal strategy in the future. The same wage rate in the two sectors may imply different values, because the two sectors have different job separation rates and arrival rates of future offers. Conversely, different wage rates may be needed in the two sectors in order to generate the same value. Since workers and firms rank offers by the value, rather than the wage rate, we will henceforth refer to the worker value V as an *offer*. Similarly, let  $V_u$  be the value for an unemployed worker. Because an unemployed worker can receive offers in addition to the unemployment benefit, then  $V_u \geq b/r$ .

On-the-job search generates a non-degenerate, continuous distribution of worker values in each sector, for the same reason as in BM. Suppose to the contrary that a mass of firms in sector *i* offer a value  $V_0 < y_i/r$ . An individual firm deviating to a slightly higher offer  $V_0 + \varepsilon$ , where  $\varepsilon > 0$ , will attract this mass of workers. The deviation is profitable because it increases the acceptance probability by a discrete amount which is more than compensating for the slightly higher offer. This contradicts the supposition that the distribution of offers is discontinuous at  $V_0 < y_i/r$  in an equilibrium.

In sector *i*, let  $F_i$  be the cumulative distribution function of values offered and  $G_i$ the cumulative distribution function of values at which workers are employed. These two distributions are endogenous and have the same support denoted  $supp_i \equiv [\underline{V}_i, \overline{V}_i]$ . The function *F* is referred to as the offer distribution, and *G* as the employed distribution. Offers are bounded below by  $\underline{V}_i \geq V_u$ , because a worker is better off staying unemployed than accepting an offer lower than  $V_u$ . Also, because wages cannot exceed output, offers are bounded above by  $y_i/r$  in sector *i*. Let  $[\underline{w}_i, \overline{w}_i]$  be the support of wage rates in sector *i*.

Assumption 1.  $F_i$  and  $G_i$  are differentiable in the interior of supp<sub>i</sub> except possibly when an interior point is on the boundary of supp<sub>-i</sub>. In addition,  $\underline{V}_A \geq \underline{V}_B$ .

<sup>&</sup>lt;sup>5</sup>For models of on-the-job search in which search is directed by firms' offers, see Delacroix and Shi (2006) and Shi (2009). For on-the-job search models that allow for wage-tenure contracts, see Burdett and Coles (2003) and Shi (2009). Postel-Vinay and Robin (2002) and Lise and Robin (2013) allow firms to compete against outside offers in a second-price auction.

Differentiability of the cumulative distributions captures the intuitive feature that a firm's tradeoff between different offers is smooth. A possible exception is an interior point in sector *i*'s support that is also a boundary point of sector -i's support, at which sector *i*'s cumulative distributions may fail to be differentiable (see Lemma 2.2).<sup>6</sup> The assumption  $\underline{V}_A \geq \underline{V}_B$  is natural since sector A has higher productivity than sector B.<sup>7</sup> Note that we do not impose  $\overline{V}_A \geq \overline{V}_B$ . Also, the supports of the two sectors' distributions may overlap.

# **Remark 1.** $\underline{V}_B = V_u$ and $\underline{V}_A \leq \overline{V}_B$ . If $\underline{V}_A \geq \overline{V}_B$ , then $\underline{V}_A = \overline{V}_B$ .

The results in this remark are intuitive. If  $\underline{V}_B > V_u$ , instead, a firm in sector B that offers  $\underline{V}_B$  can profit by deviating to a slightly lower offer. This deviation is profitable because the offer has the same rate of acceptance as the offer  $\underline{V}_B$  but yields higher profit if the vacancy is filled. Similarly, with the focus on  $\underline{V}_A \ge \underline{V}_B$ , it must be the case that  $\underline{V}_A \le \overline{V}_B$ . If  $\underline{V}_A > \overline{V}_B$ , instead, a firm offering  $\underline{V}_A$  in sector A can profit by deviating to an offer between  $\overline{V}_B$  and  $\underline{V}_A$ . This deviating offer has the same rate of acceptance as the offer  $\underline{V}_A$  has (i.e., by unemployed workers and by those employed in sector B) but yields higher ex post profit, which cannot be an equilibrium.

The equilibrium has two possible configurations, as depicted in Figure 1. The only difference between the two configurations is that  $\bar{V}_A \ge \bar{V}_B$  in configuration 1 but  $\bar{V}_A < \bar{V}_B$  in configuration 2. In configuration 1, it is possible that  $\underline{V}_A = \bar{V}_B$  or  $\underline{V}_A = \underline{V}_B$ . In configuration 2, it is possible that  $\underline{V}_A = \underline{V}_B$ . Note that even though  $\underline{V}_A \ge \underline{V}_B$  by assumption, the equilibrium may have  $\underline{w}_A < \underline{w}_B$ . A worker may be willing to start at a low wage rate in sector A if sector A has a higher job-to-job transition rate and/or a larger room for wage growth. In general, a job with a higher value is a better job for a worker, regardless of which sector the job is located, but a job with a higher wage rate may not necessarily be a better job, depending on the sector in which the job is located.

<sup>&</sup>lt;sup>6</sup>In the one-sector BM model, one can prove that the distributions are differentiable in the interior of the support by arguing that a firm's expected profit of a vacancy must be differentiable. This proof can break down in a two-sector model, because the two sectors' distributions can be non-differentiable in particular ways without making a firm's expected profit non-differentiable.

<sup>&</sup>lt;sup>7</sup>Sections 4 and 5 will specify the restrictions on the parameters for the equilibrium to have  $\underline{V}_A \geq \underline{V}_B$ . The analysis can be modified for the case with  $\underline{V}_A < \underline{V}_B$ .

This contrast between the value and the wage rate illustrates one of the advantages of formulating the equilibrium in terms of values rather than wage rates as in BM.



Figure 1. Equilibrium configurations

### 2.2. Workers' Value Functions and Optimal Choices

A worker accepts an offer if and only if the offer is higher than or equal to the value of the worker's current position. We characterize the steady-state value functions of workers when the equilibrium has  $\bar{V}_A \geq \bar{V}_B$ . Straightforward modifications of the equations yield the value functions in the case  $\bar{V}_A < \bar{V}_B$ .

Start with an unemployed worker. Such a worker gets the unemployment benefit, b, and receives an offer from sector i at the rate  $\alpha_i$ . Since an unemployed worker accepts all  $V \ge V_u$ , the value  $V_u$  obeys the following Bellman equation:

$$rV_u = b + \alpha_A \int_{\underline{V}_A}^{\overline{V}_A} (x - V_u) \, dF_A(x) + \alpha_B \int_{\underline{V}_B}^{\overline{V}_B} (x - V_u) \, dF_B(x).$$

The term  $rV_u$  is the "permanent income" associated with the present value  $V_u$ . The righthand side of the equation is the sum of the flow of income, b, and expected "capital gains" in the future. Integrating by parts and using the result  $\underline{V}_B = V_u$ , we rewrite

$$rV_{u} = b + \alpha_{A} \left[ \underline{V}_{A} - V_{u} + \int_{\underline{V}_{A}}^{\overline{V}_{A}} [1 - F_{A}(x)] dx \right] + \alpha_{B} \int_{\underline{V}_{B}}^{\overline{V}_{B}} [1 - F_{B}(x)] dx.$$
(2.1)

The term  $(\underline{V}_A - V_u)$  appears here because  $1 - F_A(x) = 1$  for all  $x \leq \underline{V}_A$ .

Consider a worker employed in sector A at a wage rate w that generates a value V to the worker, where  $V \in supp_A = [\underline{V}_A, \overline{V}_A]$ . The worker receives an outside offer from sector A at the rate  $\lambda_A$  and from sector B at the rate  $\sigma_B$ . An offer x from either sector is accepted iff  $x \geq V$ . If  $V \in [\underline{V}_A, \overline{V}_B)$ , there is positive probability that an acceptable offer comes from either sector. On the other hand, if  $V \in [\bar{V}_B, \bar{V}_A]$ , no offer from sector *B* dominates the worker's current employment. Taking into account the exogenous separation rate,  $\delta_A$ , we can derive the Bellman equation for the worker value *V* in sector *A* as

$$rV = w + \lambda_A \int_V^{\overline{V}_A} [1 - F_A(x)] dx + \sigma_B \int_V^{\overline{V}_B} [1 - F_B(x)] dx + \delta_A (V_u - V)$$
  
if  $V \in [\underline{V}_A, \overline{V}_B];$  (2.2)

$$rV = w + \lambda_A \int_V^{V_A} [1 - F_A(x)] dx + \delta_A (V_u - V) \quad \text{if } V \in (\bar{V}_B, \bar{V}_A]$$

Similarly, consider a worker employed in sector B at a wage rate w that yields the present value  $V \in supp_B = [\underline{V}_B, \overline{V}_B]$ . This worker accepts an offer x if and only if  $x \geq V$ , regardless of whether the offer comes from sector A or B. We separate the case where  $V \in [\underline{V}_B, \underline{V}_A)$  from the case where  $V \in [\underline{V}_A, \overline{V}_B]$ , because integration by parts yields different expressions in the two cases. The Bellman equation for V in sector B is:

$$rV = w + \lambda_B \int_V^{\bar{V}_B} [1 - F_B(x)] dx + \sigma_A \left[ \underline{V}_A - V + \int_{\underline{V}_A}^{\bar{V}_A} [1 - F_A(x)] dx \right] + \delta_B (V_u - V) \quad \text{if } V \in [\underline{V}_B, \underline{V}_A);$$

$$rV = w + \lambda_B \int_V^{\bar{V}_B} [1 - F_B(x)] dx + \sigma_A \int_V^{\bar{V}_A} [1 - F_A(x)] dx + \delta_B (V_u - V) \quad \text{if } V \in [\underline{V}_A, \bar{V}_B].$$

$$(2.3)$$

It is useful to rewrite (2.2) and (2.3) by keeping the wage rate on one side and moving all other terms to the other side of the equation. Doing so yields the wage rate w as a function of V. Denote this function in sector i as  $w_i(V)$  and referred to it as the wage function in sector i. For any V, the wage function in sector i specifies the wage rate that is needed to deliver V in sector i. The wage function is sector specific because the same value may require different wage rates to deliver in the two sectors.

The wage function is continuously differentiable, as shown from (2.2) and (2.3):

$$w'_i(V) = r + s_i(V) \text{ for all } V \ge V_u, \tag{2.4}$$

where  $s_i(V)$  is the separation rate from a job at value V in sector i and is defined as

$$s_i(V) \equiv \delta_i + \lambda_i \left[1 - F_i(V)\right] + \sigma_{-i} \left[1 - F_{-i}(V)\right] \text{ for all } V \ge V_u.$$

$$(2.5)$$

The separation rate in sector i is the sum of the rate of endogenous separation to another job in sector i,  $\lambda_i [1 - F_i(V)]$ , endogenous separation to a job in the other sector -i,  $\sigma_{-i} [1 - F_{-i}(V)]$ , and exogenous separation,  $\delta_i$ . The effective discount rate in sector *i* is  $(r + s_i)$ , because job separation terminates the flow of wage income and profit of match. Thus, (2.4) states intuitively that to increase the present value for a worker by a marginal unit, the wage rate must increase by the effective discount rate. Moreover, because the density functions are non-negative, then  $s'_i(V) \leq 0$ , with strict inequality if  $F'_A(V) > 0$  or  $F'_B(V) > 0$ . A higher offer reduces separation.

#### 2.3. Firms' Value Functions and Optimal Choices

A firm chooses the sector in which to post a vacancy and the offer to make. Since a firm can post only one vacancy at a time and the total measure of firms is fixed at one, the total measure of vacancies in the economy is one at every instant. Thus, we normalize the vacancy cost to zero without loss of generality. However, the distributions of vacancies over offers within each sector and between the two sectors are endogenously determined by the requirement that expected profit should be the same for all vacancies.

Suppose that a firm in sector i offers  $V \in supp_i$ . Three groups of workers accept this offer. First, unemployed workers always accept the offer. Since the measure of vacancies in sector i is  $m_i$ , the rate at which the particular firm's offer reaches unemployed worker is  $\alpha_i u/m_i$ . Second, workers who are employed in sector i at values lower than or equal to V accept the offer. The measure of such workers is  $n_iG_i(V)$ , where  $n_i$  is the measure of employed workers in sector i and  $G_i$  is the employed distribution function in sector i. The rate at which the particular firm's offer is accepted by a worker employed in sector iis  $n_iG_i(V)\lambda_i/m_i$ . Third, workers who are employed in sector -i at values lower than or equal to V accept the offer. The rate of such acceptance is  $n_{-i}G_{-i}(V)\sigma_i/m_i$ . Therefore, a firm offering V in sector i fills a vacancy at the rate  $h_i(V)/m_i$ , where

$$h_i(V) \equiv \alpha_i u + \lambda_i n_i G_i(V) + \sigma_i n_{-i} G_{-i}(V), \quad \text{for all } V \ge V_u.$$

$$(2.6)$$

Note that  $h'_i(V) \ge 0$ , with strict inequality if the employed distribution in either sector has strictly positive density at V. That is, a higher offer increases the hiring rate.

If the firm fills a vacancy, the flow of profit is  $[y_i - w_i(V)]$ . Because the effective

discount rate in sector *i* is  $[r + s_i(V)]$ , the filled job generates the present value of profits,  $\frac{y_i - w_i(V)}{r + s_i(V)}$ . The expected profit of a vacancy offering *V* in sector *i* is

$$\hat{\pi}_i(V) \equiv \left[\frac{y_i - w_i(V)}{r + s_i(V)}\right] \frac{h_i(V)}{m_i}.$$
(2.7)

The firm chooses the offer to maximize  $\hat{\pi}_i(V)$ . The optimal offer makes the tradeoff similar to that in BM. A higher offer reduces the flow of profits because it requires a higher wage rate. However, a higher offer has the benefits in recruiting and retention. A higher offer increases the hiring rate, which increases expected profit of a vacancy. In addition, because a higher offer reduces the separation rate, it reduces the effective discount rate and, for any given flow of profits, increases the present value of profits.

There is a continuum of offers that all make the optimal tradeoff between the cost of the wage and the benefits in recruiting and retention. In sector *i*, this continuum is the interval,  $supp_i$ , on which the distributions  $F_i$  and  $G_i$  are formed. Let  $\pi_i$  denote the maximized expected profit of a vacancy in sector *i*. All values outside the support must achieve strictly lower expected profit than  $\pi_i$ . Thus,

$$\hat{\pi}_i(V) = \pi_i \text{ for all } V \in supp_i, \text{ and } \hat{\pi}_i(V) < \pi_i \text{ for all } V \notin supp_i.$$
 (2.8)

It is useful to write (2.8) in a differential form. Consider any  $V > V_u$ . Expected profit  $\hat{\pi}_i(V)$  is differentiable in the interior of  $supp_i$ , because  $w_i(V)$ ,  $s_i(V)$  and  $h_i(V)$  are so. Since  $\hat{\pi}_i(V)$  is constant for all  $V \in supp_i$ , then  $\hat{\pi}'_i(V) = 0$  for such  $V \in (\underline{V}_i, \overline{V}_i)$ . Let  $\varepsilon > 0$  be sufficiently small. If  $V = \underline{V}_i - \varepsilon$ , then  $\hat{\pi}_i(V) < \hat{\pi}_i(\underline{V}_i)$ , and so  $\hat{\pi}'_i(V) > 0$ . If  $V = \overline{V}_i + \varepsilon$ , then  $\hat{\pi}_i(V) < \hat{\pi}_i(\underline{V}_i)$ , and so  $\hat{\pi}'_i(V)$ , substituting  $w'_i$  with (2.4) and taking the limit  $\varepsilon \downarrow 0$ , we express this implication of (2.8) as

$$\frac{h'_{i}(V)}{h_{i}(V)} + \frac{[-s'_{i}(V)]}{r + s_{i}(V)} - \frac{h_{i}(V)}{\hat{\pi}_{i}(V)m_{i}} \begin{cases} = 0, & \text{if } V \in (\underline{V}_{i}, \overline{V}_{i}) \\ \ge 0, & \text{if } V = \underline{V}_{i}^{-} \\ \le 0, & \text{if } V = \overline{V}_{i}^{+}. \end{cases}$$
(2.9)

Here,  $V^-$  denotes an offer arbitrarily close to V on the left, and  $V^+$  an offer arbitrarily close to V on the right. The two inequalities are in the weak form because the limit does not necessarily preserve strict inequalities. The three terms on the left-hand side are proportional benefits and cost of a higher value offer. The first term is the proportional benefit in recruiting, and the second term is the proportional benefit in retention. The third term is the proportional reduction in the profit flow caused by a higher wage rate, where we substituted  $w'_i$  using (2.4) and  $(y - w_i)$  using (2.7).

Now consider a firm's choice of the sector in which to create a vacancy. The firm chooses the sector that yields higher expected profit of a vacancy. If the two sectors co-exist, which is our focus, expected profit of a vacancy must be the same in the two sectors. Let the common expected profit be  $\pi$ . The optimal choice between the two sectors implies

$$\pi_i = \pi \text{ for } i = A, B. \tag{2.10}$$

**Remark 2.** There is a close link between a firm's optimization here and that in BM, despite that we do not model firm size as BM do. BM focus on the limit  $r \to 0$ . They reason that a firm in this limit maximizes total profit,  $[y - w(V)] \ell(V)$ , where  $\ell$  is firm size given by  $\ell(V) = \frac{nG'(V)}{mF'(V)}$ . In the limit  $r \to 0$  in our model,  $\hat{\pi}(V) = [y - w(V)] \frac{h(V)}{s(V)m}$  (see (2.7)) which coincides with that in BM because  $\frac{h(V)}{s(V)} = \frac{nG'(V)}{F'(V)}$  as established in (2.15).

#### 2.4. Steady State Flows

For every equilibrium value V, the flow of workers into V must be equal to the flow out of V in the steady state. For brevity, we analyze steady state flows for the configuration with  $\bar{V}_A \geq \bar{V}_B$ . Equilibrium values lie in  $[\underline{V}_B, \bar{V}_A]$ . We partition this interval into  $[\underline{V}_B, \underline{V}_A)$ ,  $[\underline{V}_A, \bar{V}_B)$  and  $[\bar{V}_B, \bar{V}_A]$ , and calculate the flows of workers for each subinterval separately.

First, for any value  $V \in [\underline{V}_B, \underline{V}_A)$ , consider the group of workers employed in  $(V, \underline{V}_A)$ . These workers are in sector B only and their measure is  $n_B[G_B(\underline{V}_A) - G_B(V)]$ . A worker in this group exits the group at the rate  $s_B(\underline{V}_A)$ . To calculate the inflow, note that all offers in the interval  $(V, \underline{V}_A)$  come from sector B. A random offer from sector B lies in  $(V, \underline{V}_A)$  with probability  $[F_B(\underline{V}_A) - F_B(V)]$ . Since an offer in this interval is accepted by unemployed workers and by the workers who are employed in sector B at values lower than or equal to V, the acceptance rate is  $h_B(V)$ , where we have used the fact that  $G_A(V) = 0$  for  $V \leq \underline{V}_A$ .<sup>8</sup> Equating the outflow of workers from the group to the inflow, we have:

$$s_B(\underline{V}_A) n_B[G_B(\underline{V}_A) - G_B(V)] = h_B(V) [F_B(\underline{V}_A) - F_B(V)] \text{ if } V \in [\underline{V}_B, \underline{V}_A], (2.11)$$

where  $s_B(\underline{V}_A) = \delta_B + \lambda_B [1 - F_B(\underline{V}_A)] + \sigma_A$  and  $h_B(V) = \alpha_B u + \lambda_B n_B G_B(V)$ .

Second, for any  $V \in [\underline{V}_A, \overline{V}_B)$ , consider the workers employed in sector A in  $(V, \overline{V}_B)$ . The measure of this group is  $n_A[G_A(\overline{V}_B) - G_A(V)]$ . The steady-state flow equation is:

$$s_{A}(\bar{V}_{B}) n_{A}[G_{A}(\bar{V}_{B}) - G_{A}(V)] + \sigma_{B}n_{A}\int_{V}^{\bar{V}_{B}} [1 - F_{B}(x)] dG_{A}(x)$$

$$= h_{A}(V) [F_{A}(\bar{V}_{B}) - F_{A}(V)] + \sigma_{A}n_{B}\int_{V}^{\bar{V}_{B}} [F_{A}(\bar{V}_{B}) - F_{A}(x)] dG_{B}(x), \quad \text{if } V \in [\underline{V}_{A}, \bar{V}_{B}).$$
(2.12)

The first term on the left-hand side is the measure of workers in the group who separate exogenously into unemployment or accept offers in sector A that are higher than or equal to  $\bar{V}_B$ . The second term on the left-hand side is the measure of workers in the group who accept offers in sector B that are higher than or equal to the current offer in sector A. Although the values for these workers remain in  $(V, \bar{V}_B)$ , they have switched the sector. The first term on the right-hand side is the measure of workers who are at values lower than V and are newly hired into the group. The second term on the right-hand side is the measure of workers who are employed in sector B in  $(V, \bar{V}_B)$  and are newly hired into the group in sector A.

Third, for any value  $V \in [\underline{V}_A, \overline{V}_B)$ , consider the group of workers employed in sector B in  $(V, \overline{V}_B)$ . The steady-state flow equation is analogous to (2.12):

$$s_{B}(\bar{V}_{B}) n_{B}[1 - G_{B}(V)] + \sigma_{A} n_{B} \int_{V}^{\bar{V}_{B}} [F_{A}(\bar{V}_{B}) - F_{A}(x)] dG_{B}(x)$$

$$= h_{B}(V) [1 - F_{B}(V)] + \sigma_{B} n_{A} \int_{V}^{\bar{V}_{B}} [1 - F_{B}(x)] dG_{A}(x) \quad \text{if } V \in [\underline{V}_{A}, \bar{V}_{B}).$$
(2.13)

Finally, for any value  $V \in [\bar{V}_B, \bar{V}_A]$ , consider the group of workers employed in  $(V, \bar{V}_A]$ . These workers are employed in sector A only and their measure is  $n_A[1-G_A(V)]$ . Following the same procedure as the above, we can derive the following equation:

$$s_A(\bar{V}_A) n_A[1 - G_A(V)] = h_A(V)[1 - F_A(V)], \text{ if } V \in [\bar{V}_B, \bar{V}_A], \qquad (2.14)$$

<sup>&</sup>lt;sup>8</sup>The rate  $h_B$  is not divided by  $m_B$  here because it is the acceptance of the offers of all firms in the group rather than of an individual firm.

where  $s_A(\bar{V}_A) = \delta_A$ , and  $h_A(V) = \alpha_A u + \lambda_A n_A G_A(V) + \sigma_A n_B$  for  $V \in [\bar{V}_B, \bar{V}_A]$ .

Differentiating (2.11) - (2.14) with respect to V, we obtain a simple and unified equation:

$$n_i G'_i(V) \, s_i(V) = h_i(V) \, F'_i(V) \text{ for all } V.$$
 (2.15)

This equation states the intuitive result that, at every value V, the measure of workers separating from V in each sector i must be equal to the measure of workers newly recruited at V. Note that this equation holds for all V, not just for V on the support of the distribution in the sector. For future use, we write this requirement equivalently as

$$\frac{\lambda_i n_i G'_i(V)}{h_i(V)} = \frac{\lambda_i F'_i(V)}{s_i(V)} \text{ for all } V.$$
(2.16)

Because  $\lambda_i n_i G'_i(V)$  is the increase in hiring from other firms in sector *i*, generated by a marginal increase in the offer, we term the ratio  $\frac{\lambda_i n_i G'_i(V)}{h_i(V)}$  the marginal intensity of withinsector hiring at *V* in sector *i*. Similarly, because  $\lambda_i F'_i(V)$  is the reduction in separation of workers to other firms in sector *i*, generated by a marginal increase in the offer, we term the ratio  $\frac{\lambda_i F'_i(V)}{s_i(V)}$  the marginal intensity of within-sector separation from *V* in sector *i*. In each sector, (2.16) requires that the marginal intensity of within-sector separation from *V*.

In contrast to within-sector marginal intensities, the marginal intensity of betweensector hiring at V by sector i from sector -i is  $\sigma_i n_{-i}G'_{-i}(V)/h_i(V)$ , and the marginal intensity of between-sector separation at V from sector i to sector -i is  $\sigma_{-i}F'_{-i}(V)/s_i(V)$ . The sum of the marginal intensities of within-sector and between-sector hiring by sector i is the marginal intensity of overall hiring at V in sector i,  $h'_i(V)/h_i(V)$ . The sum of the marginal intensities of within-sector and between-sector separation from sector i is the marginal intensities of within-sector and between-sector separation from sector i is the marginal intensities of within-sector hiring and separation in a sector should be equal to each other at every value, the marginal intensities of between-sector hiring and separation in a sector are not necessarily equal to each other. Neither are the marginal intensities of overall hiring and separation in a sector.

### 2.5. The Definition of the Steady State

We define the steady state of the economy as follows:

**Definition 2.1.** The steady state of the economy consists of the measure of unemployed workers, u, the measure  $(n_i)$  and the distribution  $(G_i)$  of workers employed in each sector i, the measure  $(m_i)$  and the distribution  $(F_i)$  of offers/vacancies in each sector i, expected profit of a vacancy  $(\pi)$ , and the value functions of workers. These variables and functions are time-invariant and satisfy the following requirements. (i) Optimal offers in each sector i satisfy (2.8). (ii) Expected profit of a vacancy is equalized between the two sectors, as required by (2.10). (iii) The value functions satisfy (2.1), (2.2), (2.3),  $\underline{V}_B = V_u$  and  $\underline{V}_A \geq \underline{V}_B$ . (iv) The measures of workers, the distributions of workers and the distributions of offers satisfy (2.11) - (2.14), with  $n_A + n_B + u = 1$ ,  $m_A + m_B = 1$ ,  $m_A > 0$  and  $m_B > 0$ .

The following remarks clarify the definition. First, we focus on the steady state where the two sectors co-exist, i.e.,  $m_A > 0$  and  $m_B > 0$ . The co-existence requires restrictions on parameters that will be specified in Propositions 4.1 and 5.1. Second, we separate (i) and (ii) for operational reasons. The equal-profit condition within each sector, (i), will be used to find the equilibrium relationships between the offer and employed distributions, for any given  $(m_A, m_B)$ . The equal-profit condition between sectors, (ii), will be used to solve for  $m_A$  and  $m_B$ . Third, it is straightforward to derive the wage rate distributions from the value distributions. For example, the distribution of wage rate offers in sector *i*, denoted as  $F_{w,i}(w)$ , obeys  $F_{w,i}(w_i(V)) = F_i(V)$ , where  $w_i(V)$  is the wage rate needed to generate the value V in sector *i*. The density function of wage rate offers in sector *i* is  $F'_{w,i}(w) = F'_i(V)/w'_i(V)$ , where  $w'_i(V)$  is given by (2.4).

Let us denote  $J = supp_A \cap supp_B$  and term J the overlapping set. If J has positive measure, the distributions in the two sectors are said to be *overlapping*, the steady state to be an overlapping steady state, and the worker values in J to be overlapping values. If Jhas zero measure, the distributions in the two sectors are said to be *non-overlapping* and the steady state to be a non-overlapping steady state. From the equilibrium configurations in Figure 1, the steady state is overlapping if and only if  $\underline{V}_A < \overline{V}_B$ . In this case, the upper bound on J is the smaller one between  $\bar{V}_A$  and  $\bar{V}_B$ . If the steady state is non-overlapping, then  $\underline{V}_A \geq \bar{V}_B$ . In this case, Remark 1 shows  $\underline{V}_A = \bar{V}_B$ , and so J contains the singleton  $\{\bar{V}_B\}$  or, equivalently,  $\{\underline{V}_A\}$ . Let  $F'(V^-)$  denote the left-hand derivative of F at V and  $F'(V^+)$  the right-hand derivative of F at V. The following lemma states these features of J and the bounds on the supports (see Appendix A for a proof):

**Lemma 2.2.**  $J = [\underline{V}_A, \min\{\overline{V}_A, \overline{V}_B\}]$ . If J has zero measure, then  $\underline{V}_A = \overline{V}_B$ . Maintain Assumption 1 and assume  $\sigma_A > 0$  and  $\sigma_B > 0$ . If  $\overline{V}_i$  lies in the interior of supp<sub>-i</sub>, then

$$F'_{-i}\left(\bar{V}_{i}^{+}\right) - F'_{-i}\left(\bar{V}_{i}^{-}\right) = X_{-i,i}\left(\bar{V}_{i}\right)F'_{i}\left(\bar{V}_{i}^{-}\right)$$
(2.17)

$$X_{i,-i}\left(\bar{V}_{i}\right)\left[F_{-i}'\left(\bar{V}_{i}^{+}\right)-F_{-i}'\left(\bar{V}_{i}^{-}\right)\right] \leq F_{i}'\left(\bar{V}_{i}^{-}\right)$$

$$(2.18)$$

where  $X_{i,j}(V) > 0$  is defined in (A.2) in Appendix A. Thus,  $F'_i(\bar{V}_i) = 0$  if and only if  $F'_{-i}(\bar{V}_i^+) = F'_{-i}(\bar{V}_i^-)$ , while  $F'_i(\bar{V}_i^-) > 0$  if and only if  $F'_{-i}(\bar{V}_i^+) > F'_{-i}(\bar{V}_i^-)$  and  $X_{-i,i}(\bar{V}_i) X_{i,-i}(\bar{V}_i) \le 1$ . Similarly, if  $\underline{V}_A$  lies in the interior of supp<sub>B</sub>, then

$$F'_{B}\left(\underline{V}_{A}^{-}\right) - F'_{B}\left(\underline{V}_{A}^{+}\right) = X_{B,A}\left(\underline{V}_{A}\right)F'_{A}\left(\underline{V}_{A}^{+}\right)$$

$$(2.19)$$

$$X_{A,B}\left(\underline{V}_{A}\right)\left[F_{B}'\left(\underline{V}_{A}^{-}\right)-F_{B}'\left(\underline{V}_{A}^{+}\right)\right] \geq F_{A}'\left(\underline{V}_{A}^{+}\right).$$
(2.20)

Thus,  $F'_A(\underline{V}_A) = 0$  if and only if  $F'_B(\underline{V}_A) = F'_B(\underline{V}_A)$ , while  $F'_A(\underline{V}_A) > 0$  if and only if  $F'_B(\underline{V}_A) > F'_B(\underline{V}_A)$  and  $X_{B,A}(\underline{V}_A) X_{A,B}(\underline{V}_A) \ge 1$ .

Lemma 2.2 shows that the cumulative distribution functions in one sector may fail to be differentiable if an interior point in the sector's support is a bound on the other sector's support.<sup>9</sup> Consider the case where  $\bar{V}_i$  lies in the interior of  $supp_{-i}$ . If  $F_i$  is differentiable at  $\bar{V}_i$ , i.e., if  $F'_i(\bar{V}_i) = 0$ , then  $F_{-i}$  is differentiable at  $\bar{V}_i$ . In this case, the tradeoff of hiring and separation against the wage cost is smooth on the two sides of  $\bar{V}_i$  for firms in the two sectors, and so (2.17) and (2.18) are both satisfied. If  $F_i$  is not differentiable at  $\bar{V}_i$ , i.e., if  $F'_i(\bar{V}_i^-) > 0$ , then  $F_{-i}$  is not differentiable at  $\bar{V}_i$ , with  $F'_{-i}(\bar{V}_i^+) > F'_{-i}(\bar{V}_i^-)$ . In this case, the marginal benefit and cost in hiring and separation change discontinuously when

<sup>&</sup>lt;sup>9</sup>(2.15) shows that  $G_i$  is not differentiable at V if and only if  $F_i$  is not differentiable at V.

the offer changes from one side of  $\bar{V}_i$  to the other side. (2.17) ensures that a firm in sector -i is indifferent between the offers on the two sides of  $\bar{V}_i$ , and (2.18) ensures that it is not profitable for a firm in sector i to offer values above  $\bar{V}_i$ .

To elaborate, suppose  $F'_i(\bar{V}_i^-) > 0$ , where  $\bar{V}_i$  lies in the interior of  $supp_{-i}$ , and let  $\varepsilon > 0$ be arbitrarily small. When a firm increases the offer from  $\bar{V}_i - \varepsilon$  to  $\bar{V}_i$ , the firm increases hiring from and reduces separation to other firms in sector i. Since (2.15) ties marginal hiring to marginal separation in the steady state, this benefit from sector i is proportional to  $F'_i(\bar{V}_i^-)$ . This benefit does not arise when a firm increases the offer from  $\bar{V}_i$  to  $\bar{V}_i + \varepsilon$ , because no workers are employed above  $\bar{V}_i$  in sector *i*. However, a firm in sector -i must be indifferent among the offers on the two sides of  $\bar{V}_i$  in order for  $\bar{V}_i$  to lie in the interior of  $supp_{-i}$ . For such indifference, the asymmetry in the benefit from sector *i* should be offset by the opposite asymmetry in the benefit from sector -i. That is, the increase from  $\bar{V}_i$  to  $\bar{V}_i + \varepsilon$  should yield a higher benefit of hiring and separation from sector -i than the increase from  $\bar{V}_i - \varepsilon$  to  $\bar{V}_i$  does. This requires  $\left[F'_{-i}\left(\bar{V}_i^+\right) - F'_{-i}\left(\bar{V}_i^-\right)\right]$  to be positive and related to  $F'_{i}(\bar{V}_{i})$  according to (2.17), where the multiplier  $X_{-i,i}(\bar{V}_{i})$  converts the between-sector benefits or losses into amounts comparable with the within-sector terms. In contrast, a firm in sector i should not profit from offering values above  $V_i$ . To ensure this, the increase in the offer from  $\bar{V}_i$  to  $\bar{V}_i + \varepsilon$  should yield a benefit from sector -i that is less than or equal to the loss from sector *i*. This requirement is (2.18), where  $X_{-i,i}(\bar{V}_i)$ is applied to the benefit from sector -i because this benefit is from the other sector for a firm in sector i.

For the lower bound  $\underline{V}_A$  to lie in the interior of  $supp_B$ , the required conditions, (2.19) and (2.20), can be explained similarly to the above, with two modifications. First, the inequality between the densities of  $F_B$  on the two sides of  $\underline{V}_A$  holds as  $F'_B(\underline{V}_A^-) \ge F'_B(\underline{V}_A^+)$ , instead of the other way around, because  $F'_A(\underline{V}_A^+) \ge F'_A(\underline{V}_A^-) = 0$ . Second, the deviation to be prevented is a downward deviation below  $\underline{V}_A$  by a firm in sector A, in contrast to an upward deviation above  $\overline{V}_i$  by a firm in sector i. This contrast explains why the inequality in (2.20) is opposite to that in (2.18). The potential non-differentiability is caused by between-sector search. If  $\sigma_A = \sigma_B = 0$ , then  $X_{-i,i}(\bar{V}_i) = X_{i,-i}(\bar{V}_i) = 0$  and  $X_{B,A}(\underline{V}_A) = X_{A,B}(\underline{V}_A) = 0$  (see (A.2)). In this case, (2.17) shows that  $F'_{-i}(\bar{V}_i)$  exists, and (2.19) shows that  $F'_B(\underline{V}_A)$  exists. Moreover, since  $X_{B,A}(\underline{V}_A) X_{A,B}(\underline{V}_A) = 0$  violates the condition for  $F'_A(\underline{V}_A) > 0$ , then  $F'_A(\underline{V}_A) = 0$ . We extend these results to sufficiently small  $(\sigma_A, \sigma_B)$ :

**Remark 3.** Suppose that  $\sigma_A$  and  $\sigma_B$  are sufficiently close to 0. If  $\bar{V}_i$  lies in the interior of  $\operatorname{supp}_{-i}$ , then  $F'_{-i}(\bar{V}_i)$  exists. If  $\underline{V}_A$  lies in the interior of  $\operatorname{supp}_B$ , then  $F'_B(\underline{V}_A)$  exists and  $F'_A(\underline{V}_A) = 0$ .

Beyond Lemma 2.2, the equilibrium is difficult to analyze even in the steady state. The main difficulty is that the flows of workers between the two sectors depend on the distributions in the two sectors, as reflected by the two integrals in (2.12) and (2.13). Although (2.15) seems a simpler alternative, it cannot be solved analytically in general, because  $h_i$  involves both  $G_i$  and  $G_{-i}$ , and  $s_i$  involves both  $F_i$  and  $F_{-i}$ . In sections 3 - 5, we will explore a property that makes the steady state tractable. When this property does not hold, we will compute numerical examples in section 6. In that case, Lemma 2.2 will provide initial conditions for solving the system of functional equations in the equilibrium.

## 3. Constant Tension and the Link to the One-Sector Model

For a tractable analysis, we explore the property that  $h_i(V) s_i(V)$  is constant for all  $V \in supp_i$ . We term this property *constant tension* in sector *i*. Since  $h_i$  and  $s_i$  are differentiable in the interior of the support, constant tension in sector *i* is equivalent to

$$\frac{h'_i(V)}{h_i(V)} = \frac{-s'_i(V)}{s_i(V)} \text{ for all } V \text{ in the interior of } supp_i.$$
(3.1)

That is, constant tension means that the marginal intensities of overall hiring and separation are balanced at every value in the interior of the support of the distribution in a sector. To see how constant tension improves tractability and to connect to the literature, we establish Lemma 3.1 and Corollary 3.2 (both are proven in Appendix A): **Lemma 3.1.** Assume  $h_i(V) s_i(V) = \tau_i$  for all  $V \in \text{supp}_i$ , where  $\tau_i > 0$  is a constant. Then

$$V - \underline{V}_{i} = \frac{m_{i}\pi_{i}}{\tau_{i}} \left\{ 2\left[s_{i}\left(\underline{V}_{i}\right) - s_{i}\left(V\right)\right] + r \ln\left[\frac{r + s_{i}\left(V\right)}{r + s_{i}\left(\underline{V}_{i}\right)}\right] \right\}.$$
(3.2)

In the limit  $r \to 0$ ,  $s_i(V)$  and  $w_i(V)$  are given by

$$s_i(V) - s_i(\underline{V}_i) = \frac{-\tau_i}{2m_i\pi_i}(V - \underline{V}_i), \qquad (3.3)$$

$$w_i(V) - \underline{w}_i = s_i(\underline{V}_i)(V - \underline{V}_i) - \frac{\tau_i}{4m_i\pi_i}(V - \underline{V}_i)^2.$$
(3.4)

The solution (3.2) is the inverse function of the separation rate in sector *i*. Constant tension leads to this explicit solution because it links all endogenous elements of the expected profit function of a vacancy exclusively to the separation rate in that sector. To see this, consider a vacancy offering V in sector *i*. Constant tension links the hiring rate of the vacancy to the reciprocal of the separation rate at V in sector *i*. Since the derivative of the wage rate implied by V in sector *i* is a linear function of the separation rate (see (2.4)), expected profit of the vacancy given by (2.7) depends only on the separation rate  $s_i(V)$  and its integral. Across all equilibrium offers in sector *i*, expected profit of a vacancy must be the same. In the form (2.9), this equilibrium requirement produces a differential equation in  $s_i(V)$ , which can be solved to yield (3.2).

It is remarkable that if there is constant tension, the separation rate is a linear function of the offer when time discounting goes to zero.<sup>10</sup> To explain how this result arises, let us suppress the subscript *i*. When time discounting vanishes, the present value of a filled job is the profit flow discounted by the separation rate. With constant tension, the hiring rate of a vacancy offering V is proportional to 1/s(V), and so expected present value of the profit of the vacancy is proportional to  $[y - w(V)]/s^2(V)$  (see (2.7)). Because such expected profit must be constant over all optimal offers, then  $y - w(V) = \rho s^2(V)$  for some constant  $\rho > 0$ . This implies  $w'(V)/s(V) = -2\rho s'(V)$ . Recall w'(V) = s(V) when  $r \to 0$ 

<sup>&</sup>lt;sup>10</sup>In this paper, "linear" means "affine". Also, the equilibrium with no time discounting means the limit  $r \to 0$  of the sequence of equilibria with r > 0. As is well known, an economy literally at r = 0 may admit more equilibria than the limit equilibrium.

(see (2.4)). Thus, when time discounting vanishes, constant tension implies that s'(V) is constant over V; i.e., s(V) is a linear function. Also, since w'(V) is linear in this case, the wage function is quadratic, as in (3.4).

To appreciate the significance of Lemma 3.1, we note that the one-sector BM model is a special case of the lemma. The following corollary states this fact and the implications:

**Corollary 3.2.** The one-sector model has the following properties. (i)  $h(V) s(V) = \tau$  for all  $V \in [\underline{V}, \overline{V}]$ , where  $\tau = \alpha u (\delta + \lambda)$ , and so s(V) is given by (3.2) without the subscript *i*. (ii) For all r > 0, the density function of value offers is strictly increasing. (iii) In the limit  $r \to 0$ , s(V) satisfies (3.3) and w(V) satisfies (3.4), with  $s(\underline{V}) = \delta + \lambda$ . In this limit, the offer distribution of values is uniform and given as

$$F(V) = \frac{\tau}{2\lambda\pi} \left( V - \underline{V} \right), \qquad (3.5)$$

where we have normalized m = 1, and the density function of wage rate offers is

$$F'_{w} = \frac{\tau}{2\lambda\pi} \left[ \left(\delta + \lambda\right)^{2} - \frac{\tau}{\pi} \left(w - \underline{w}\right) \right]^{-1/2}.$$
(3.6)

In the one-sector BM, the steady state has constant tension because there is no distinction between within-sector and overall hiring or separation. Since the marginal intensities of within-sector hiring and separation are balanced at every worker value (see (2.16)), the marginal intensities of overall hiring and separation are balanced at every value automatically. The latter balance is constant tension (see (3.1)). The wage density function (3.6) is well known from BM. However, it has remained largely unknown in the literature that offers in the worker value are uniformly distributed in the limit  $r \to 0$ .

The uniform distribution of value offers, although striking, comes from the linear separation rate generated by constant tension in the limit  $r \rightarrow 0$ . It is useful to explain why value offers are more evenly spread out than wage offers. A higher wage rate enables a firm to fill a vacancy more quickly and to retain the worker for a longer time. Because of these benefits of a high wage rate to a firm, more firms must compete at high wages than at low wages in order to equalize expected profit of a vacancy across all equilibrium wage offers. That is, the density of wage offers is an increasing function, as is well known in the BM. However, because a worker separates less at higher wage rates, the same increase in the wage rate translates into a larger increase in the worker value when the wage rate is high than when the wage rate is low. In this sense, value offers more evenly spread out. In the limit  $r \rightarrow 0$ , the density of value offers becomes completely flat.<sup>11</sup>

Constant tension is not unique to the one-sector model. The following proposition describes the cases of constant tension with two sectors (see Appendix B for a proof):

**Proposition 3.3.** Assume that the steady state has constant tension in the two sectors. If  $\sigma_A = \sigma_B = 0$ , the steady state is necessarily overlapping. If  $\sigma_A > 0$  or  $\sigma_B > 0$ , the steady state is non-overlapping except possibly for a measure zero set of parameter values that satisfy  $\lambda_A > \sigma_A$ ,  $\lambda_B > \sigma_B$ ,  $\sigma_A = \sigma_B$  and  $\delta_A - \delta_B = \lambda_B - \lambda_A$ .

The case  $\sigma_A = \sigma_B = 0$  will be analyzed in section 4 and the case with non-overlapping support in section 5. If  $\sigma_A > 0$  or  $\sigma_B > 0$ , constant tension requires the steady state to be non-overlapping, except possibly for a measure zero set of parameter values that satisfy all of the conditions in Proposition 3.3. To see why this result arises, suppose that an overlapping steady state with constant tension exists with positive between-sector search rates. We explain why this supposition is inconsistent with the steady state.

First, the two sectors' separation rates must be proportional to each other at all overlapping values. At an overlapping value, constant tension requires the marginal intensities of overall hiring and separation to be equal to each other in each sector. Because the marginal intensities of within-sector hiring and separation are equal to each other, then the marginal intensities of between-sector hiring and separation are also equal to each other at every overlapping value. For sector A, this requirement is  $\sigma_A n_B G'_B(V) / h_A(V) = \sigma_B F'_B(V) / s_A(V)$ for all overlapping values. Since V is also offered in sector B, steady state flows within sector B require  $n_B G'_B(V) / h_B(V) = F'_B(V) / s_B(V)$ . These two requirements are both satisfied only if  $s_A(V) = \psi s_B(V)$  for all overlapping values, where  $\psi > 0$  is a constant.

<sup>&</sup>lt;sup>11</sup>In this case, the employed density of worker values is still increasing.

Second,  $\psi = 1$ , and so  $s_A(V) = s_B(V)$  for all overlapping values. A firm must be indifferent between all offers in the overlapping set and between the two sectors. Specifically, (2.9) represents the local indifference between different offers in sector *i*. With constant tension, all three terms on the left-hand side of (2.9) are related exclusively to the separation rate. If  $\psi \neq 1$ , a change in *V* in the overlapping set would change some of these terms in different proportions between the two sectors, in which case (2.9) would not hold in both sectors on all overlapping values (see Appendix B for the details). Thus,  $\psi = 1$ must hold. Note that this further implies  $s'_A(V) = s'_B(V)$  for all overlapping values and so  $m_A \pi_A / \tau_A = m_B \pi_B / \tau_B$  (see (3.2)), where  $\tau_i$  is the constant tension in sector *i*.

Third, the offer densities in both sectors are strictly positive in the interior of the overlapping set J. Since the separation rate in J is equal between the two sectors, -s' is the common determinant of the offer densities in the two sectors in J. The solution (3.2) shows that -s'(V) > 0 in the interior of J. Thus,  $F'_A(V)$  and  $F'_B(V)$  are also strictly positive in the interior of J, which remain strictly positive when V approaches the bounds of J from within J. Since  $F'_A(\underline{V}^+_A) > 0$ , Lemma 2.2 implies that  $F'_B$  is discontinuous at  $\underline{V}_A$  in the form  $F'_B(\underline{V}^-_A) > F'_B(\underline{V}^+_A)$ . Moreover, if  $\overline{V}_B < \overline{V}_A$ , then  $F'_A$  is discontinuous at  $\overline{V}_B$  in the form  $F'_A(\overline{V}^+_B) > F'_A(\overline{V}^-_B)$ , because  $F'_B(\overline{V}^-_A) > 0$ . If  $\overline{V}_A < \overline{V}_B$ , then  $F'_B$  is discontinuous at  $\overline{V}_A$  in the form  $F'_B(\overline{V}^+_A) > F'_B(\overline{V}^-_A)$ , because  $F'_A(V^-_A) > 0$ .

Finally, these features implied by constant tension on the overlapping set are inconsistent with equal profitability of a vacancy between the two sectors. For brevity, we explain the inconsistency in the case  $\lambda_A \lambda_B > \sigma_A \sigma_B$ , and refer the other cases to Appendix B. In this case,  $\lambda_A > \sigma_A$  and  $\lambda_B > \sigma_B$  must hold. If  $\lambda_A \leq \sigma_A$ , for example, then  $\lambda_B > \sigma_B$ , in which case sector A poaches workers from sector B at a faster rate than sector B does from sector A, and so sector B will not exist in the steady state. Suppose  $\lambda_A > \sigma_A$  and  $\lambda_B > \sigma_B$ . With the relatively small between-sector search rates, a firm in one sector obtains a relatively small benefit from the other sector by increasing the offer. In this case, the two conditions for  $\underline{V}_A$  to lie in the interior of  $supp_B$ , (2.19) and (2.20), cannot both be satisfied. For a firm in sector B to be indifferent among the offers on the two sides of  $\underline{V}_A$ , the benefit to a firm in sector B generated by increasing an offer from  $\overline{V}_i - \varepsilon$  to  $\overline{V}_i$  must be relatively small, but in this case it is profitable for a firm in sector A to deviate to offers below  $\underline{V}_A$ . Since  $\underline{V}_A$  must coincide with  $\underline{V}_B$ , the requirements  $s_A(\underline{V}_A) = s_B(\underline{V}_B)$  and (2.15) lead to  $\delta_A - \delta_B = \lambda_B - \lambda_A$  and  $\sigma_A = \sigma_B$ , which can only be satisfied by a measure zero set of parameter values.

## 4. Between-Sector Search Only by Unemployed Workers

In this section we impose the restrictions  $\sigma_A = \sigma_B = 0$  so that an unemployed worker can search in both sectors, but an employed worker can search on the job only in the worker's incumbent sector. These restrictions force any sectoral transitions of workers to be interrupted by a spell of unemployment. Despite these restrictions, the case encompasses all studies of multi-sectoral search models without on-the-job search as special cases. The latter impose the additional restrictions  $\lambda_A = \lambda_B = 0$ , an example of which is Beaudry et al. (2012). In contrast, with  $\lambda_A > 0$  and  $\lambda_B > 0$ , employed workers' search in their incumbent sector directly affects expected profit of a vacancy in each sector and, hence, the distribution of vacancies both within each sector and between the two sectors. This equilibrium effect affects all workers' transitions and creates a link between the entire sector-specific wage distributions, not just the average wage rates. The proofs of all results in this section are in Appendix C.

With  $\sigma_A = \sigma_B = 0$ , the separation rate of a worker from a firm in sector *i* depends only on the offer distribution in sector *i* and not on the offer distribution in the other sector. Also, the hiring rate of a firm in sector *i*,  $h_i/m_i$ , depends only on the employed distribution in sector *i* and not on the employed distribution in the other sector. These features of hiring and separation imply that the marginal intensities of overall hiring and separation are equal to each other at every value in each sector. That is, there is constant tension. The following proposition states this result and determines the steady state:

**Proposition 4.1.** Assume  $\sigma_A = \sigma_B = 0$  and denote  $\theta_i = \delta_i / \lambda_i$ . Then,  $h_i(V) s_i(V) = \tau_i$  for all  $V \in \text{supp}_i$ , where  $\tau_i = \alpha_i u (\delta_i + \lambda_i)$ , and so  $s_i(V)$  is given by (3.2). Moreover,  $\underline{V}_A = \underline{V}_B (= V_u)$ , and the two sectors' distributions necessarily overlap. Consider the limit

 $r \rightarrow 0$ . The steady state with two active sectors exists if and only if

$$\frac{-\theta_B\left(\theta_B+2\right)}{\theta_B\left(\theta_B+2\right)+\alpha_B/\lambda_B} < \frac{y_A - y_B}{y_B - b} < \frac{\lambda_A}{\alpha_A}\theta_A\left(\theta_A+2\right).$$

$$(4.1)$$

The offer and employed distributions in sector i are

$$F_i(V) = \frac{\alpha_i u\left(\theta_i + 1\right)}{2m_i \pi_i} \left(V - \underline{V}_i\right), \quad G_i(V) = \theta_i \left[\frac{2m_i \pi_i}{\alpha_i u\left(V - \underline{V}_i\right)} - 1\right]^{-1}, \tag{4.2}$$

and the composition of workers in the economy is:

$$u = \frac{\delta_A \delta_B}{\delta_A \delta_B + \alpha_A \lambda_B + \alpha_B \lambda_A}, \quad n_i = \frac{\alpha_i}{\delta_i} u, \quad i \in \{A, B\}.$$
(4.3)

The supports of the distributions have the same lower bound, i.e.,  $\underline{V}_A = \underline{V}_B$ . The explanation for this result is similar to that for  $\underline{V}_B = V_u$  in Remark 1: The case  $\underline{V}_A > \underline{V}_B$  would present a profitable opportunity for a firm to offer a worker value in  $(\underline{V}_B, \underline{V}_A)$ . However, the argument relies on the assumption in the current case that employed workers cannot search in the other sector.<sup>12</sup>

For the two sectors to co-exist, the productivity differential between the two sectors cannot be too large or too small, as in (4.1). If sector A is much more productive than sector B, all vacancies will be located in sector A and, with positive exogenous separation, no worker will be in sector B in the steady state. Similarly, if sector B is much more productive than sector A, a case ruled out by assumption, all vacancies will be located in sector B and no worker will be in sector A in the steady state. Equal productivity between the two sectors is consistent with the co-existence of the two sectors in the steady state. Also, the bounds on the productivity differential depend on all parameters that affect workers' mobility rates in the two sectors, i.e.,  $(\alpha_i, \delta_i, \lambda_i)$  for  $i \in \{A, B\}$ .

The unemployment rate depends on all mobility parameters. With (4.3), it is easy to see that the unemployment rate is lower if the exogenous separation rate is lower, if the arrival rate of an offer to an unemployed worker is higher, or if the arrival rate of an offer to an employed worker is higher. In contrast, the relative size of employment of sector

<sup>&</sup>lt;sup>12</sup>This result holds even without the maintained assumption  $\underline{V}_A \geq \underline{V}_B$ . If  $\underline{V}_B > \underline{V}_A$ , contrary to the assumption, then a firm offering  $\underline{V}_B$  in sector B can profit from reducing the offer slightly, which cannot be an equilibrium.

A to sector B,  $n_A/n_B$ , depends only on the two ratios,  $\alpha_A/\delta_A$  and  $\alpha_B/\delta_B$ . The higher is the ratio  $\alpha_i/\delta_i$ , the larger is the relative size of sector *i* to the other sector. Job-to-job transition rates affect the size of a sector, but they do not affect the relative size between the two sectors since there is no between-sector search on the job.

Note that an unanticipated one-time increase in the relative productivity  $y_A/y_B$  can have a dramatic effect on sectoral employment shares, even if the increase is small. Suppose for example that  $y_A/y_B$  satisfies the bound in (4.1) by an amount  $\varepsilon$  before the one-time shock and violates the bound afterwards by an amount  $\varepsilon$ . Then the employment share of sector *B* will drop discontinuously from a positive number to zero. The reason is that steady-state sectoral employment shares depend on frictional parameters only, but not on the productivity differential, as is required by the steady-state flow equations.

We now characterize a number of properties of the wage- and value distributions. Define the average wage rate in sector i as  $\mathbb{E}w_i \equiv \int_{\underline{V}_i}^{\underline{V}_i} w_i(V) dG_i(V)$ . The following proposition compares values and wage rates between the two sectors:

**Proposition 4.2.** Assume  $\sigma_A = \sigma_B = 0$ . In the limit  $r \to 0$ , the results (i)-(iv) hold: (i)  $\bar{w}_A > \bar{w}_B$  if and only if

$$\frac{y_A - y_B}{y_B - b} > \frac{\theta_A \theta_B \left(\theta_A - \theta_B\right)}{\theta_A \theta_B \left(\theta_B + 2\right) + \theta_A \alpha_B / \lambda_B + \theta_B \alpha_A / \lambda_A}.$$
(4.4)

(ii)  $\underline{w}_A < \underline{w}_B$  if and only if

$$\frac{y_A - y_B}{y_B - b} > \frac{(\theta_A + \theta_B + 2)(\theta_A - \theta_B)}{\theta_B(\theta_B + 2) + \alpha_A/\lambda_A + \alpha_B/\lambda_B}.$$
(4.5)

The lower bound in (4.5) is greater than that in (4.4) if and only if  $\theta_A > \theta_B$ .

(iii)  $\mathbb{E}w_A - \mathbb{E}w_B = \frac{1}{2}(\bar{w}_A - \bar{w}_B)$ , and so  $\mathbb{E}w_A > \mathbb{E}w_B$  if and only if (4.4) holds.

(iv)  $F_A$  first-order dominates  $F_B$  if and only if  $\bar{V}_A > \bar{V}_B$ , which is equivalent to

$$\frac{y_A - y_B}{y_B - b} > \frac{\lambda_A \theta_A \left(\theta_A + 2\right) - \lambda_B \theta_B \left(\theta_B + 2\right)}{\lambda_B \theta_B \left(\theta_B + 2\right) + \alpha_A + \alpha_B}.$$
(4.6)

When  $\theta_A = \theta_B$ ,  $G_A$  first-order stochastically dominates  $G_B$  if and only if (4.6) holds.

The comparison between the supports of the two sectors' distributions depends critically on the productivity differential between the two sectors and the ratio  $\theta_i = \delta_i / \lambda_i$ . Because this ratio depends on the arrival rate of an offer to an employed worker, on-the-job search affects the relative level of and relative dispersion in wage rates and values between the two sectors. This effect of on-the-job search is a general equilibrium effect. Although an employed worker can only search in the worker's incumbent sector, such on-the-job search affects firms' decisions on which sector to locate a vacancy and what offers to make. As a result, the distributions of wage rates and worker values in one sector depend on the on-the-job search parameter  $\lambda$  in the other sector.

To understand various conditions in Proposition 4.2, it is useful to consider first the case  $\theta_A = \theta_B$ . In this case, the bounds on the productivity differential in (4.4) and (4.5) are both zero. Since these two conditions are satisfied, the highest wage rate in sector A is higher than that in sector B, and the lowest wage rate in sector A is lower than in sector B. In this sense, higher productivity increases dispersion in wage rates in sector A relative to sector B. This effect of productivity on wage dispersion is intuitive. When productivity is higher in sector A, competition among firms pushes the highest wage rate in the sector above that in sector B. Expecting this larger room for the wage rate to grow in sector A, an unemployed worker is willing to accept a lower wage rate to start in sector A than in sector B. Despite a lower starting wage rate in sector A, the lifetime income of starting at the lowest wage rate is the same in the two sectors, as given by  $\underline{V}_A = \underline{V}_B$ .

The larger dispersion in wage rates in sector A relative to sector B is not a meanpreserving spread. In the case  $\theta_A = \theta_B$ , in particular, the average wage rate is higher in sector A than in sector B. The between-sector differential in the average wage rate is a half of the between-sector differential in the highest wage rate.<sup>13</sup>

It is remarkable that these comparisons between the two sectors in the case  $\theta_A = \theta_B$  are independent of  $\alpha_A$  and  $\alpha_B$ , the arrival rates of offers from the two sectors to unemployed workers. The arrival rates  $\alpha_A$  and  $\alpha_B$  affect the levels of wage rates in each sector, but

<sup>&</sup>lt;sup>13</sup>Although the density function of the wage distribution in each sector is increasing, the overall distribution of wage rates in the economy can have different shapes. For example, when the support of sector B's distribution is contained as a subset of the support of sector A's distribution, the density function of the overall distribution of wage rates may increase first, have a drop at the upper bound of sector B's support, and then increasing again.

they do not affect whether the support of the wage rate distribution is wider in one sector than in the other sector, provided  $\theta_A = \theta_B$ .

The comparison between the two sectors' wage rates does not necessarily carry over to the comparison between the present values. We have already explained that the lowest worker value is the same in the two sectors. However, whether the highest worker value in sector A is higher than that in sector B depends on the difference between the two sectors' on-the-job search parameter  $\lambda$ , as well as on the ratio  $\theta$ . If the highest value in sector A exceeds that in sector B, then value offers are higher in sector A than in sector B in the sense of first-order order stochastic dominance, because the distribution of values is uniform in each sector. This result holds independently of whether  $\theta_A$  is equal to  $\theta_B$ .

In the case  $\theta_A = \theta_B$ , sector A has higher offers than sector B in the sense of first-order stochastic dominance if and only if the productivity differential between the two sectors is large relative to the differential between the two sectors' arrival rates of offers to employed workers. A sufficient condition for this dominance is  $\lambda_A \leq \lambda_B$ . The same condition is sufficient for employed workers to have higher values in sector A than in sector B in the sense of first-order stochastic dominance. To explain these results, note that in the case  $\theta_A = \theta_B$ ,  $\lambda_A \leq \lambda_B$  is equivalent to  $\delta_A \leq \delta_B$ . Thus, if  $\lambda_A \leq \lambda_B$  and if the two sectors have the same distribution, then an employed worker separates from a job no more likely in sector A than in sector B. Given higher productivity in sector A, a firm can make higher profit in sector B. If  $\lambda_A > \lambda_B$ , the same result occurs when the productivity differential between the two sectors dominates the between-sector difference in the separation rate.

To further understand the importance of search frictions, we compare the two sectors when  $\theta_A \neq \theta_B$ . If  $\theta_A < \theta_B$ , most of the above comparisons between the two sectors are still valid. If  $\theta_A > \theta_B$ , the above comparisons hold only if the difference  $(\theta_A - \theta_B)$  is small relative to the productivity differential between the two sectors, as given by (4.4) - (4.6). To explain why these conditions are needed, note that the case  $\theta_A > \theta_B$  can be interpreted as one where on-the-job search frictions are more severe in sector A than in sector B, because the offer arrival rate to employed workers relative to the exogenous separation rate is lower in sector A than in sector B. The productivity differential must dominate this differential in on-the-job search frictions in order for sector A to have a larger highest wage rate and a larger average wage rate than sector B.<sup>14</sup>

We can also compare the two sectors' worker transition rates. With the distributions in (4.2), the average job-to-job transition rate in sector *i* is

$$\int_{\underline{V}_{i}}^{V_{i}} \lambda_{i} \left[1 - F_{i}\left(V\right)\right] dG_{i}\left(V\right) = \lambda_{i} \theta_{i} \left[\left(\theta_{i} + 1\right) \ln\left(1 + \frac{1}{\theta_{i}}\right) - 1\right].$$

The expression inside [.] is a strictly decreasing function of  $\theta_i$ . Recall  $\theta_i = \delta_i/\lambda_i$ . If the two sectors have the same exogenous job separation rate, then a sector has a higher average rate of job-to-job transitions if and only if the offer arrival rate to employed workers is higher in that sector. Also, the product of  $\theta_i$  and the above expression in [.] is a strictly increasing function of  $\theta_i$ . Thus, holding  $\lambda$  to be the same in the two sectors, the sector with a higher exogenous separation rate also has a higher average rate of job-to-job transitions. The reason is that with a higher exogenous separation rate, the job-to-job transition rate must be higher in order to keep the distribution of employed workers over values stationary.

Finally, we examine how the unemployment benefit affects the steady state:<sup>15</sup>

**Corollary 4.3.** Assume  $\sigma_A = \sigma_B = 0$ . An increase in the unemployment benefit *b* increases  $(\bar{w}_A - \bar{w}_B)$ , increases  $(\mathbb{E}w_A - \mathbb{E}w_B)$ , and reduces  $(\underline{w}_A - \underline{w}_B)$ , if and only if  $\theta_A > \theta_B$ . Such an increase in *b* increases  $(\bar{V}_A - \bar{V}_B)$  if and only if  $\lambda_A \theta_A (\theta_A + 2) > \lambda_B \theta_B (\theta_B + 2)$ .

If  $\theta_A = \theta_B$ , the unemployment benefit has no effect on the spread in wage rates. If  $\theta_A \neq \theta_B$ , the effect of the unemployment benefit on wage rates is not uniform between the two sectors. If  $\theta_A < \theta_B$ , the support of wage rates in sector B is contained in the support in sector A. In this case, an increase in the unemployment benefit reduces the wage spread in sector A relative to sector B by reducing the gap between the two sectors in both the

<sup>&</sup>lt;sup>14</sup>When  $\theta_A > \theta_B$ , the offer arrival rates to unemployed workers,  $(\alpha_A, \alpha_B)$ , affect the bounds in (4.4) - (4.6) and, hence, affect the spread in values in one sector relative to the other sector.

<sup>&</sup>lt;sup>15</sup>An increase in b has no effect on  $(u, n_A, n_B)$  because the total number of vacancies is fixed. If there is free entry of vacancies, then an increase in b will increase u by reducing the profitability of a vacancy.

highest and the lowest wage rate. If  $\theta_A > \theta_B$ , an increase in the unemployment benefit increases the wage spread in sector A relative to sector B. In particular, if the support of wage rates in sector B is still contained in the support in sector A, an increase in the unemployment benefit in the case  $\theta_A > \theta_B$  makes the highest wage rate even higher and the lowest wage rate even lower in sector A relative to sector B. Recall that on-the-job search frictions are more severe in sector A than in sector B if and only if  $\theta_A > \theta_B$ . Thus, an increase in the unemployment benefit increases the wage spread in the sector with more severe frictions in on-the-job search relative to the other sector.

In contrast to wage rates, the spread in worker values can increase in b by more in sector A than in sector B even if  $\theta_A = \theta_B$ . This is the case if  $\lambda_A > \lambda_B$ .

## 5. Non-Overlapping Distributions with Between-Sector Search

In this section we allow all workers to search in both sectors, but restrict the parameters in such a way that the distributions in the two sectors do not overlap. Recall that  $\underline{V}_A = \overline{V}_B$  in the non-overlapping steady state (see Lemma 2.2). Proposition 3.3 has shown that if the steady state has constant tension with  $\sigma_A > 0$  and  $\sigma_B > 0$ , then it is non-overlapping generically. The following proposition shows that the reverse is also true and characterizes the non-overlapping steady state (see Appendix D for a proof):

**Proposition 5.1.** If the steady state is non-overlapping, then it has constant tension, and so the separation rate is given by (3.2). Focus on the limit  $r \to 0$ . A non-overlapping steady state with two active sectors exists if and only if

$$\frac{y_A - y_B}{y_B - b} > \max\{k_1, 0\}, \quad \frac{y_B - b}{y_A - y_B} > \max\{k_2, 0\}, \tag{5.1}$$

$$k_3(y_A - y_B) + k_4(y_B - b) \ge 0, \tag{5.2}$$

where  $(k_1, k_2, k_3, k_4)$  are constants defined in Appendix D. If (but not only if)  $\alpha_A \geq \sigma_A$ , there exists a non-empty set of values of  $(y_A, y_B)$  that satisfy (5.1) and (5.2). The sufficient conditions for  $\underline{w}_A < \overline{w}_B$  are  $\delta_A = \delta_B$  and  $\sigma_A < \lambda_A$ . Moreover, an increase in the unemployment benefit b reduces  $(\overline{w}_A - \underline{w}_A)$  and  $(\overline{V}_A - \underline{V}_A)$  if and only if  $2\lambda_B (\delta_A - \delta_B) < 22$ 

 $(\delta_B + \sigma_A)^2$ . An increase in *b* always reduces  $(\bar{w}_B - \underline{w}_B)$  and  $(\bar{V}_B - \underline{V}_B)$ , and increases  $(\bar{w}_A - \underline{w}_A) / (\bar{w}_B - \underline{w}_B)$  and  $(\bar{V}_A - \underline{V}_A) / (\bar{V}_B - \underline{V}_B)$ .

When the two sectors' distributions do not overlap, the steady state has constant tension because the between-sector flow of employed workers is one directional: from sector B to sector A. Because an offer in sector B is accepted only by employed workers in sector B and unemployed workers, the hiring rate of a vacancy in sector B depends only on sector B's, but not on sector A's, employed distribution. Thus, the marginal intensity of hiring within sector B is equal to the marginal intensity of overall hiring in sector B at every value. Also, because all offers in sector A are accepted by all workers employed in sector B, the hiring rate of a vacancy in sector A depends on sector A's employed distribution but not on sector B's employed distribution. Although the measure of workers employed in sector B affects the hiring rate in sector A, this effect is the same on all offers in sector A. Thus, the marginal intensity of hiring within sector A is equal to the marginal intensity of overall hiring in sector A at every value. Similarly, the marginal intensity of within-sector separation from a worker value is equal to the marginal intensity of overall separation from that value in the same sector. In this case, (2.16) implies that the marginal intensities of overall hiring and separation are equal to each other at every value in a sector. This equality is equivalent to constant tension (see (3.1)).

The conditions in (5.1) are necessary and sufficient for the two sectors to co-exist. Not surprisingly, these conditions require the productivity differential between the two sector to be neither too large nor too small. Too large a differential in productivity will eliminate sector B, and too small a differential productivity will eliminate sector A. However, the conditions in (5.1) are not sufficient for the non-overlapping steady state to exist – (5.2) is also needed.<sup>16</sup> This additional condition makes it not profitable for a firm in sector A to deviate to an offer slightly below  $\underline{V}_A$  or for a firm in sector B to deviate to an offer slightly above  $\overline{V}_B$ . Although (5.2) restricts the productivity differential further, in general it cannot be written in the format of (5.1), because neither  $k_3$  nor  $k_4$  is necessarily positive.

<sup>&</sup>lt;sup>16</sup>Recall that we have focused on the steady state with  $\underline{V}_A \ge \underline{V}_B$ . The violation of (5.2) does not rule out the existence of a non-overlapping steady state with  $\underline{V}_A = V_u < \underline{V}_B$ .

If  $k_3 > 0$ , then (5.2) becomes  $\frac{y_A - y_B}{y_B - b} > \frac{-k_4}{k_3}$ , which is stronger than the first part of (5.1). Regardless of the sign of  $k_3$ , there exists a non-empty region of the parameters in which (5.1) and (5.2) are both satisfied.

Although worker values in the two sectors do not overlap under (5.1) and (5.2), wage rates can overlap. Specifically, if  $\delta_A = \delta_B$  and  $\sigma_A < \lambda_A$ , then the lowest wage rate in sector A is lower than the highest wage in sector B. These sufficient conditions for overlapping wage rates are intuitive. Consider two workers both employed at the wage rate  $\bar{w}_B$ , one in sector A and the other in sector B. For both workers, the only upward mobility is to move to firms in sector A that offer higher wage rates. If  $\sigma_A < \lambda_A$ , the worker employed in sector A has higher upward mobility than the worker employed in sector B. If  $\delta_A = \delta_B$ , in addition, the two workers face the same rate at which their current job is destroyed exogenously. Thus, the value for the worker employed in sector A at  $\bar{w}_B$  must be strictly higher than the value for the worker employed in sector B at the same wage rate. This means that for  $\underline{V}_A = \bar{V}_B$  to hold as is required in the non-overlapping steady state, the wage rate  $\underline{w}_A$  that delivers the value  $\underline{V}_A$  in sector A must be strictly lower than  $\bar{w}_B$ .

Finally, an increase in the unemployment benefit b affects the spread in wage rates and the spread in values in the two sectors. An increase in b increases wages, reduces the profitability of a vacancy in both sectors, and suppresses the spread in wage rates and the spread in values in the sector closer to unemployed workers in values, namely, sector B. This squeezing effect of a higher b transmits into sector A through competition among firms. If the exogenous separation rate in sector A does not exceed that in sector B by too much, the spread in wage rates and the spread in values also become narrower in sector A. However, this squeezing effect of b is not uniform between the two sectors. Because productivity is lower in sector B than in sector A, the profit margin is also smaller in section B. As an increase in b increases wages, the profit margin falls by a larger proportion in section B than in sector A. Although the measure of vacancies falls in both sectors, it falls by more in sector B, which compresses the spread in wage rates in sector B by more than in sector A. This increases the ratio of the wage spread in sector A to sector B. Proposition 5.1 characterizes the non-overlapping steady state generally. To illustrate such a steady state, we turn to a specific case with the following restrictions:

$$\frac{\alpha_A}{\alpha_B} = \frac{\lambda_A}{\sigma_B} = \frac{\sigma_A}{\lambda_B} = a, \quad \frac{\delta_A}{\delta_B} = \frac{\lambda_A}{\sigma_A} = c \quad \left(=\frac{\sigma_B}{\lambda_B}\right), \tag{5.3}$$

where a > 0 and c > 0 are arbitrary constants. Note that these require  $\sigma_A > 0$  and  $\sigma_B > 0$ . The first set of restrictions in (5.3) requires that, for all workers, the arrival rate of an offer from sector A should be *a* times that from sector B. The ratio *a* is independent of where a worker is located in the economy, i.e., whether a worker is unemployed and in which sector a worker is employed. Of course, the offer arrival rate in each sector can still depend on where a worker is. The second set of restrictions in (5.3) requires that the exogenous separation rate in sector A relative to sector B should be equal to the relative rate at which an employed worker in sector A, versus sector B, receives an offer from sector A. Note that (5.3) implies  $\theta_A = \theta_B/a$ , where  $\theta_i = \delta_i/\lambda_i$  for i = A, B. According to the explanation in section 4, a higher  $\theta$  implies more severe frictions in on-the-job search. Thus, frictions in on-the-job search are less severe in sector A than in sector B if and only if a > 1.

The special case c = 1 is easier to explain. In this case, the second set of restrictions in (5.3) becomes  $\delta_A = \delta_B$ ,  $\sigma_A = \lambda_A$ , and  $\sigma_B = \lambda_B$ . A worker separates exogenously into unemployment at the same rate in the two sectors, an employed worker receives an offer from sector A at the same rate independently of which sector the worker is employed, and an employed worker receives an offer from sector B at the same rate independently of which sector the worker is employed. The only differences between the two sectors are the differentials in productivity and in the rate at which a sector's offer reaches a worker.

For arbitrary constants a > 0 and c > 0, (5.3) yields

$$\frac{h_A(V)}{h_B(V)} = a, \quad \frac{s_A(V)}{s_B(V)} = c \quad \text{for all } V.$$

The marginal intensity of overall hiring at any value,  $h'_i(V)/h_i(V)$ , is equal between the two sectors, and so is the marginal intensity of overall separation at any value,  $-s'_i(V)/s_i(V)$ . The difference between these two rates, which is the marginal intensity of churning in a sector, must be equal between the two sectors at all values. However, the marginal intensities of churning must sum up to zero between the two sectors in order to maintain

the steady state. This is possible if and only if the marginal intensity of churning is zero in each sector. Since the latter result is equivalent to constant tension (see (3.1)), (5.3)implies constant tension.<sup>17</sup>

With constant tension (and  $\sigma_A, \sigma_B > 0$ ), Proposition 3.3 implies that the steady state is non-overlapping. We summarize this result and the implications of Proposition 5.1 in this case in the following corollary (the proof is omitted):

**Corollary 5.2.** Assume (5.3). The steady state has constant tension and is non-overlapping. Such a non-overlapping steady state with two active sectors (and with  $\underline{V}_A \geq \underline{V}_B$ ) exists if and only if

$$\frac{y_B - b}{y_A - y_B} > \max\{k_2, 0\}, \quad \frac{y_A - y_B}{y_B - b} \ge \max\{\frac{-k_4}{k_3}, \varepsilon\},$$
(5.4)

where  $\varepsilon > 0$  is an arbitrarily small number and<sup>18</sup>

$$k_{2} = \frac{c\alpha_{A}/\lambda_{A} - 1}{c\theta_{A}(\theta_{A} + 2) + 1}, \quad \frac{-k_{4}}{k_{3}} = \frac{(c - 1)\theta_{A}[a(\theta_{A} + 2) + 2]}{\theta_{A}[a(\theta_{A} + 2) + 2] + (a + 1)^{2}c\alpha_{A}/\lambda_{A}}.$$
(5.5)

All of these conditions are satisfied if (but not only if)  $c \leq 1$  and  $\alpha_A \leq \lambda_A/c$ .

Two clarifications are useful. First, since (5.3) is sufficient for constant tension and nonoverlapping distributions, the additional conditions in (5.4) are imposed not for generating these features. Rather, the first condition in (5.4) is for the two sectors to co-exist. The second condition in (5.4) is for  $\underline{V}_A \geq \underline{V}_B$  and, if this condition is violated, then the steady state will still be non-overlapping but will have  $\underline{V}_A = V_u < \underline{V}_B$ . Second, if a = c = 1, the two sectors are symmetric except productivity. Such symmetry is sufficient for (5.3)and hence for the steady state to be non-overlapping, but it is not necessary. Finally, when c = 1, the non-overlapping steady state with two active sectors exists for all values of  $a \in (0,\infty)$ . Even if the arrival rate of an offer from sector A is lower than from sector B (i.e., a < 1), sector A provides higher worker values than sector B. In this case, the

<sup>&</sup>lt;sup>17</sup>To prove this result formally, differentiate (2.6) for i = A, substitute (2.15), and use  $\sigma_A = \lambda_A/c$  to get  $h'_A/h_A = \lambda_A \left(\frac{1}{s_A}F'_A + \frac{h_B/h_A}{cs_B}F'_B\right)$ . Since  $h_A = ah_B$  and  $s_A = cs_B$ , then  $h'_A/h_A = \lambda_A \left(F'_A + \frac{1}{a}F'_B\right)/s_A$ . Differentiating (2.5) for i = A and using  $\sigma_B = \lambda_A/a$ , we get  $s'_A/s_A = -h'_A/h_A$ . Similarly,  $s'_B/s_B = -h'_B/h_B$ . <sup>18</sup>The first condition in (5.1) is not needed in this special case because it is implied by (5.2).

measure of firms in sector A may be smaller than in sector B to ensure expected profit of a vacancy to be the same in the two sectors.

## 6. Non-Constant Tension and Between-Sector Search

In this section, we examine an economy with non-constant tension. By Proposition 5.1, non-constant tension necessarily requires two-directional flows of employed workers between the two sectors. Since these flows make it difficult to analyze the steady state in general, we compute some examples. The following parameters are fixed in the numerical examples:

$$r = 0.05, \ \alpha_A = \alpha_B = 0.074, \ \delta_A = 0.111, \ \delta_B = 0.128, \ \lambda_A = \lambda_B = 0.0101, \ y_A = 1.05, \ y_B = 1, \ b = 0.3y_A.$$

Although the search parameters are loosely calibrated to the empirical transition rates in the Survey of Program Participations, they should be taken as a suggestive instead of a definitive representation of the data.

Example:	1	2	3	4
$\sigma_A$	0	0.003	0.006	0.002
$\sigma_B$	0	0.003	0.006	0.006

We explore the effect of between-sector search on the equilibrium by changing the parameters,  $\sigma_A$  and  $\sigma_B$ , as in Examples 1-4. Example 1 has constant tension but Examples 2-4 do not. For each combination of the parameter values, we compute the solution to the system of ordinary differential equations formed by (2.9) and (2.15), with the equilibrium restrictions on the bounds of the supports given by Lemma 2.2 and Remark 3.<sup>19</sup>

Figure 2 depicts the offer distributions in the two sectors in Example 1. These distributions are approximately linear because Example 1 has constant tension and a small r: when  $r \to 0$ , these functions should be exactly linear. As in section 4, the two sectors have the same lowest value offer. The highest value offered in sector A is strictly higher than the highest value offered in sector B. Moreover, offers in sector A first-order stochastically dominate those in sector B. This between-sector ranking extends the results in (iv) in Proposition 4.2 to a case with positive and small time discounting.

 $<sup>^{19}\</sup>mathrm{A}$  description of the computation procedure is available upon request.



Figure 2. Offer distributions in Example 1:  $\sigma_A = \sigma_B = 0$ 



Figure 3. Offer distributions in Example 2:  $\sigma_A = \sigma_B = 0.003$ 

Figure 3 depicts the offer distributions in Example 2. Similar to Example 1, the two sectors have the same lowest offer, and offers in sector A first-order stochastically dominate those in sector B. The offer distribution is still close to a uniform distribution in sector B, but not so in sector A. The slope of the cumulative distribution of offers in sector A is increasing, which implies that the density of offers in sector A is an increasing function. Because high offers in sector A can attract more workers from both sectors, more firms make high offers than low offers in sector A so as to ensure that expected profit is equalized across all equilibrium offers.

Figures 4 and 5 contrast the offer distributions in Example 2 with Example 1. An increase in  $\sigma$  increases offers in both sectors in first-order stochastic dominance. This is

intuitive. When employed workers can search between sectors at a higher rate, competition for workers intensifies among the firms, which increases the offers in both sectors. Note that the offer distribution in sector A shifts to the right by more than that in sector B.



Figure 4. Offer distribution in sector A with  $\sigma = 0$  and  $\sigma = 0.003$ 



Figure 5. Offer distribution in sector B with  $\sigma = 0$  and  $\sigma = 0.003$ 

Table 2 lists other features of the equilibria. First, an increase in the between-sector search rate  $\sigma$  reduces the fraction of vacancies in sector B,  $m_B$ , reduces the measure of workers in sector B, and increases the measure of workers in sector A. A relatively small increase in  $\sigma$  in both sectors, from 0 in Example 1 to 0.006 in Example 3, reduces the fraction of workers employed in sector B by 22% from 0.430 to 0.335. The measure of unemployed workers does not change much in the four examples. Second, the two sectors have approximately the same lowest wage rate in the four examples, with the lowest wage rate in sector A being slightly lower than that in sector B, but the highest wage rate in sector A is higher than that in sector B. Similarly, the two sectors have approximately the same lowest worker value in the four examples, but the highest value in sector A is higher than that in sector B. Thus, sector A has a larger spread in both wage rates and values than sector B. This result extends the possibilities in (i) and (ii) in Proposition 4.2 from an economy with  $\sigma_A = \sigma_B = 0$  to an economy with small  $\sigma_A > 0$  and  $\sigma_B > 0$ .

Example	1	2	3	4
$\sigma_A$	0	0.003	0.006	0.006
$\sigma_B$	0	0.003	0.006	0.002
$m_B$	0.463	0.461	0.460	0.454
$n_A$	0.496	0.530	0.592	0.574
$n_B$	0.430	0.397	0.335	0.352
u	0.075	0.074	0.074	0.074
$\underline{w}_A$	0.592	0.610	0.620	0.614
$\bar{w}_A$	0.885	0.927	0.956	0.925
$\underline{w}_B$	0.596	0.613	0.623	0.610
$\bar{w}_B$	0.839	0.865	0.860	0.891
$\underline{V}_A$	12.30	12.79	13.16	12.88
$\bar{V}_A$	16.72	17.50	18.04	17.48
$\underline{V}_B = V_u$	12.30	12.78	13.15	12.87
$\bar{V}_B$	15.87	16.39	16.42	16.81

Table 2. Features of the steady state

Third, an increase in the between-sector search rate  $\sigma$  increases the spread in wage rates by more in sector A than in sector B. Similarly, an increase in the between-sector search rate  $\sigma$  increases the spread in values by more in sector A than in sector B. In this sense, an increase in between-sector search increases inequality in sector A relative to sector B. This is consistent with the empirical evidence in Hoffmann and Shi (2011). Hence, our model can generate an inequality enhancing effect in the growing sector even without worker or firm heterogeneity. Finally, the contrast between the last two columns in Table 2 shows that when search between the sectors is easier for sector A workers than for sector B workers, the spreads in wage rates and values decrease in sector A and increase in sector B.

# 7. Concluding Remarks

This paper has characterized the steady state of a two-sector economy with on the job search and explored a property called constant tension. If the steady state has constant tension, the separation rate in each sector can be solved explicitly. When time discounting vanishes, this solution becomes a linear function of the worker value. The linear separation rate may be of independent interest. Because offer distributions are not observed, the linear separation rate enables one to solve offer distributions easily, which can simplify the computation and estimation of the model.

Constant tension relies on the steady state flow equations heavily, which can be a limiting factor for analyzing dynamics. For the dynamics in the one-sector BM model, the literature has made significant progress by introducing sufficient heterogeneity in productivity among firms and/or allowing firms to match offers in a second-price auction (e.g., Moscarini and Postel-Vinay, 2013; Lise and Robin, 2013). A promising alternative is to model search as a directed process (e.g., Shi, 2009; Menzio and Shi, 2011), as opposed to undirected search in the BM model. More specifically, Hoffmann and Shi (2011) have used a directed search model to account for the stylized facts on sectoral reallocation. It remains to be checked how the long-run predictions of the two-sector BM model differ from those of the models of directed search.

# Appendix

# A. Proofs of Lemma 2.2, Lemma 3.1 and Corollary 3.2

In Lemma 2.2, the result  $J = [\underline{V}_A, \min\{\overline{V}_A, \overline{V}_B\}]$  follows from the configurations in Figure 1. If J has zero measure, then  $\underline{V}_A \ge \overline{V}_B$ . In this case, Remark 1 shows  $\underline{V}_A = \overline{V}_B$ .

Assuming  $\sigma_A > 0$  and  $\sigma_B > 0$ , we prove the remainder of Lemma 2.2. Suppose that  $\bar{V}_i$ lies in the interior of  $supp_{-i}$ . Consider the choice of a firm in sector -i between the offer  $\bar{V}_i^-$  and the offer  $\bar{V}_i^+$ . Because  $\bar{V}_i$  lies in the interior of  $supp_{-i}$ , both  $\bar{V}_i^-$  and  $\bar{V}_i^+$  satisfy the first-order condition in sector -i with equality. That is, with the subscript -i instead of i on the functions, (2.9) holds with equality for  $V = \bar{V}_i^-$  and  $V = V_i^+$ . Subtracting the condition for these two offers yields:

$$\frac{h'_{-i}\left(\bar{V}_{i}^{-}\right) - h'_{-i}\left(\bar{V}_{i}^{+}\right)}{h_{-i}\left(\bar{V}_{i}\right)} - \frac{s'_{-i}\left(\bar{V}_{i}^{-}\right) - s'_{-i}\left(\bar{V}_{i}^{+}\right)}{r + s_{-i}\left(\bar{V}_{i}\right)} = 0, \tag{A.1}$$

where we have used the fact that  $h_{-i}$  and  $s_{-i}$  are continuous functions. Because  $F_i(V) = G_i(V) = 0$  for all  $V > \overline{V}_i$ , then  $F'_i(\overline{V}_i^+) = G'_i(\overline{V}_i^+) = 0$ . Using these results and differentiating (2.6) for sector -i, we get

$$h'_{-i}\left(\bar{V}_{i}^{+}\right) = \lambda_{-i}n_{-i}G'_{-i}\left(\bar{V}_{i}^{+}\right) = \frac{\lambda_{-i}h_{-i}\left(\bar{V}_{i}\right)}{s_{-i}\left(\bar{V}_{i}\right)}F'_{-i}\left(\bar{V}_{i}^{+}\right),$$

where the second equality comes from (2.15). Similarly,

$$h_{-i}'\left(\bar{V}_{i}^{-}\right) = \frac{\lambda_{-i}h_{-i}\left(\bar{V}_{i}\right)}{s_{-i}\left(\bar{V}_{i}\right)}F_{-i}'\left(\bar{V}_{i}^{-}\right) + \frac{\sigma_{-i}h_{i}\left(\bar{V}_{i}\right)}{s_{i}\left(\bar{V}_{i}\right)}F_{i}'\left(\bar{V}_{i}^{-}\right)$$

Differentiating (2.5) yields:

$$s'_{-i}\left(\bar{V}_{i}^{-}\right) = -\lambda_{-i}F'_{-i}\left(\bar{V}_{i}^{-}\right) - \sigma_{i}F'_{i}\left(\bar{V}_{i}^{-}\right), \quad s'_{-i}\left(\bar{V}_{i}^{+}\right) = -\lambda_{-i}F'_{-i}\left(\bar{V}_{i}^{+}\right).$$

Substituting these derivatives into (A.1) yields (2.17) where

$$X_{-i,i}(V) \equiv \left[\sigma_i + \frac{\sigma_{-i}h_i(V)[r + s_{-i}(V)]}{s_i(V)h_{-i}(V)}\right] \frac{1}{\lambda_{-i}} \left(2 + \frac{r}{s_{-i}(V)}\right)^{-1}.$$
 (A.2)

For  $\bar{V}_i$  to be the upper bound on the support of  $F_i$ , it must be the case that the firstorder condition for a sector *i* firm, (2.9), is satisfied with equality at  $V = \bar{V}_i^-$  and with inequality " $\leq$ " at  $V = \bar{V}_i^+$ . Subtracting the condition for the two cases yields

$$\frac{h_i'\left(\bar{V}_i^-\right) - h_i'\left(\bar{V}_i^+\right)}{h_i\left(\bar{V}_i\right)} - \frac{s_i'\left(\bar{V}_i^-\right) - s_i'\left(\bar{V}_i^+\right)}{r + s_i\left(\bar{V}_i\right)} \ge 0.$$
(A.3)
  
42

Computing the derivatives  $h'_i$  and  $s'_i$ , and substituting into (A.3), we obtain (2.18).

Under the hypothesis  $\sigma_A > 0$  and  $\sigma_B > 0$ , it is easy to verify that  $X_{-i,i}(V) > 0$  and  $X_{i,-i}(V) > 0$ . Then, (2.17) shows that  $F'_i(\bar{V}_i^-) = 0$  only if  $F'_{-i}(\bar{V}_i^+) = F'_{-i}(\bar{V}_i^-)$ . Since the features  $F'_{-i}(\bar{V}_i^+) = F'_{-i}(\bar{V}_i^-)$  and  $F'_i(\bar{V}_i^-) = 0$  satisfy (2.18), then  $F'_i(\bar{V}_i^-) = 0$  if and only if  $F'_{-i}(\bar{V}_i^+) = F'_{-i}(\bar{V}_i^-)$ . Similarly, for  $F'_i(\bar{V}_i^-) > 0$ , (2.17) and (2.18) are both satisfied if and only if  $F'_{-i}(\bar{V}_i^+) > F'_{-i}(\bar{V}_i^-)$  and  $X_{-i,i}(\bar{V}_i) X_{i,-i}(\bar{V}_i) \leq 1$ .

If  $\underline{V}_A$  lies in the interior of  $supp_B$ , we can replace  $\overline{V}_i$  with  $\underline{V}_A$ , set i = A in the above proof, and notice that (2.9) holds with inequality " $\geq$ " at  $\underline{V}_A^-$  for i = A. Since  $F'(\underline{V}_A^-) = G'_A(\underline{V}_A^-) = 0$ , the same procedure as the above leads to (2.19) and (2.20). From these conditions we can deduce that  $F'_A(\underline{V}_A^+) = 0$  if and only if  $F'_B(\underline{V}_A^-) = F'_B(\underline{V}_A^+)$ , while  $F'_A(\underline{V}_A^+) > 0$  if and only if  $F'_B(\underline{V}_A^-) > F'_B(\underline{V}_A^+) > F'_B(\underline{V}_A^-) > F'_B(\underline{V}_A^-) > F'_B(\underline{V}_A^-) > 1$ . This completes the proof of Lemma 2.2.

For Lemma 3.1, suppose  $h_i(V) s_i(V) = \tau_i$  for all  $V \in supp_i$ , where  $\tau_i > 0$  is a constant. Consider any arbitrary  $V \in supp_i$ . Then, (2.9) holds as equality, and  $\hat{\pi}_i(V) = \pi_i$  by (2.8). Substituting  $h_i = \tau_i/s_i$  into (2.9), we get:

$$\left(2 - \frac{r}{r + s_i(V)}\right) s_i'(V) = \frac{-\tau_i}{m_i \pi_i}.$$
(A.4)

Integration yields (3.2). In the limit  $r \to 0$ , (3.2) implies (3.3). Also, since (2.4) implies  $w'_i(V) = s_i(V)$  when  $r \to 0$ , integration yields (3.4). This proves Lemma 3.1.

To prove Corollary 3.2, suppose that there is only one sector and normalize m = 1. The subscript *i* is irrelevant in this case. With only one sector, (2.5) and (2.6) become:

$$s(V) = \delta + \lambda [1 - F(V)], \quad h(V) = \alpha u + \lambda n G(V).$$

Differentiating these equations yields  $s'(V) = -\lambda F'(V)$  and  $h'(V) = \lambda n G'(V)$ . Then,

$$(hs)' = h's + s'h = \lambda [nG's - hF'] = 0,$$

where the last equality follows from (2.15). Thus,  $h(V) s(V) = \tau$  for all V, where  $\tau > 0$  is a constant, which shows that the one-sector economy satisfies the hypothesis in Lemma 3.1. Then, s(V) satisfies (3.2), without the subscript i and with m = 1. Moreover, setting  $V = \underline{V}$  reveals  $\tau = \alpha u (\delta + \lambda)$ . This is property (i) in the corollary. For property (ii), differentiate (3.2), or simply use  $h = \tau/s$  in (2.9), to get  $s' = \frac{-\tau(r+s)}{\pi(r+2s)} < 0$ . Differentiating s' yields  $s''(V) = \frac{r\tau s'}{(r+2s)^2} < 0$  for all r > 0. Thus,  $F''(V) = -\frac{s''(V)}{\lambda} > 0$  for all r > 0.

For property (iii), consider the limit  $r \to 0$  in the one-sector economy. Then, s(V) satisfies (3.3) and w(V) satisfies (3.4), with  $s(\underline{V}) = \delta + \lambda$ . Because  $s(V) = \delta + \lambda [1 - F(V)]$ ,

(3.3) implies (3.5). Moreover, given the wage rate, (3.4) is a quadratic equation of  $(V - \underline{V})$ . Solve this equation for  $(V - \underline{V})$ , and take the smaller one of the two solutions to ensure s(V) > 0. Then,  $F'_{w} = F'(V) / w'(V)$  yields (3.6). **QED** 

# **B.** Proof of Proposition 3.3

The following Lemma will be used in the proof of Proposition 3.3:

**Lemma B.1.** Assume that the steady state has constant tension, with  $h_i(V) s_i(V) = \tau_i$ for all  $V \in \text{supp}_i$ , and that the distributions in the two sectors overlap on the set J of positive measure specified in Lemma 2.2. If  $s_A(V) = \psi s_B(V)$  for all  $V \in J$  and some constant  $\psi > 0$ , then  $\psi = 1$  and the following results hold for all  $V \in J$ : (i)  $s_A(V) = s_B(V)$ ; (ii)  $m_A \pi_A / \tau_A = m_B \pi_B / \tau_B$ ; (iii)  $w_A(V) - w_B(V) = y_A - y_B$ , and (iv)  $y_{A} - y_{B} = \begin{cases} (\sigma_{A} - \lambda_{A}) \int_{\bar{V}_{B}}^{\bar{V}_{A}} [1 - F_{A}(x)] dx + (\delta_{A} - \delta_{B}) (\bar{V}_{B} - V_{u}), & \text{if } \bar{V}_{A} \ge \bar{V}_{B} \\ (\lambda_{B} - \sigma_{B}) \int_{\bar{V}_{A}}^{\bar{V}_{B}} [1 - F_{B}(x)] dx + (\delta_{A} - \delta_{B}) (\bar{V}_{A} - V_{u}), & \text{if } \bar{V}_{A} < \bar{V}_{B}. \end{cases}$ (B.1)

**Proof.** Maintain the hypotheses in the lemma. Also, let  $s_A(V) = \psi s_B(V)$  for all  $V \in J$  and some constant  $\psi > 0$ , where J has positive measure. Consider any  $V \in J$ . Recall that constant tension implies (A.4). Dividing (A.4) for i = A by the same equation for i = B and using  $s_A(V) = \psi s_B(V)$ , we obtain

$$\frac{m_B \pi_B \tau_A}{\psi m_A \pi_A \tau_B} = \frac{[r + 2s_A(V)] [\psi r + s_A(V)]}{[r + s_A(V)] [\psi r + 2s_A(V)]}.$$
(B.2)

The left-hand side of (B.2) is constant over V. For all r > 0, if  $\psi \neq 1$ , the right-hand side of (B.2) varies with V, in which case (B.2) cannot hold for all  $V \in J$ . Thus,  $\psi = 1$  for all r > 0. Since the steady state with no time discounting is interpreted as the limit  $r \to 0$  of the sequence of steady states with r > 0, then  $\psi = 1$  in the limit  $r \to 0$ .

With  $\psi = 1$ ,  $s_A(x) = s_B(x)$  for all  $x \in J$ , as stated in (i) of the lemma. Also, (B.2) becomes  $m_B \pi_B / \tau_B = m_A \pi_A / \tau_A$ , as stated in (ii) of the lemma. Because  $\tau_i = h_i(V) s_i(V)$ for i = A, B, and  $s_A(V) = s_B(V)$  for all  $V \in J$ , then (ii) implies  $m_B \pi_B / h_B(V) =$  $m_A \pi_A / h_A(V)$ . With the equilibrium feature  $\hat{\pi}_i(V) = \pi_i$ , (2.7) yields

$$y_i - w_i(V) = [r + s_i(V)] \frac{m_i \pi_i}{h_i(V)}.$$

Because the right-hand side of this equation is independent of i, the left-hand side must be independent of i. That is,  $w_A(V) - w_B(V) = y_A - y_B$  for all  $V \in J$ , as stated in (iii) of the lemma. Moreover, because  $w'_{i}(V) = r + s_{i}(V)$ , i = A, B, integration yields

$$w_i(V) = w_i(\underline{V}_A) + \int_{\underline{V}_A}^{V} [r + s_i(x)] dx \text{ for all } V \in J.$$

Subtracting the versions of this equation for i = A and for i = B and using (i), we get

$$w_A(V) - w_B(V) = w_A(\underline{V}_A) - w_B(\underline{V}_A) \text{ for all } V \in J.$$
(B.3)

To establish (iv) of the lemma, we consider first the case  $\bar{V}_A \geq \bar{V}_B$  and then the case  $\bar{V}_A < \bar{V}_B$ . In the case  $\bar{V}_A \geq \bar{V}_B$ , the overlapping set of the two sectors' distributions is  $J = [\underline{V}_A, \bar{V}_B]$ . Consider any  $V \in J$ . Using the definition of  $s_i$  in (2.5), we rewrite (2.2) and (2.3) for any overlapping worker value as

$$rV = w_A(V) + \lambda_A \int_{\bar{V}_B}^{\bar{V}_A} [1 - F_A(x)] dx + \int_V^{\bar{V}_B} s_A(x) dx - \delta_A (\bar{V}_B - V_u)$$
  
$$rV = w_B(V) + \sigma_A \int_{\bar{V}_B}^{\bar{V}_A} [1 - F_A(x)] dx + \int_V^{\bar{V}_B} s_B(x) dx - \delta_B (\bar{V}_B - V_u).$$

Subtracting the two equations, and using (B.3) and  $s_A(x) = s_B(x)$  for all  $x \in J$ , we get

$$w_A\left(\underline{V}_A\right) - w_B\left(\underline{V}_A\right) = \left(\sigma_A - \lambda_A\right) \int_{\bar{V}_B}^{V_A} \left[1 - F_A\left(x\right)\right] dx + \left(\delta_A - \delta_B\right) \left(\bar{V}_B - V_u\right).$$

In the case  $\bar{V}_A \geq \bar{V}_B$ , (B.1) follows from this equation, (B.3), and (iii).

In the case  $\bar{V}_A < \bar{V}_B$ , the overlapping set of the distributions is  $J = [\underline{V}_A, \bar{V}_A]$ . In this case, the Bellman equations for any overlapping value V can be written as

$$rV = w_A(V) + \sigma_B \int_{\bar{V}_A}^{\bar{V}_B} [1 - F_B(x)] dx + \int_V^{\bar{V}_A} s_A(x) dx - \delta_A (\bar{V}_A - V_u)$$
  
$$rV = w_B(V) + \lambda_B \int_{\bar{V}_A}^{\bar{V}_B} [1 - F_B(x)] dx + \int_V^{\bar{V}_A} s_B(x) dx - \delta_B (\bar{V}_A - V_u).$$

The same procedure as the above establishes (B.1) in the case. **QED** 

### **Proof of Proposition 3.3**:

According to the hypotheses in the proposition, let  $h_i(V) s_i(V) = \tau_i$  for all  $V \in supp_i$ , i = A, B. If  $\sigma_A = \sigma_B = 0$ , the steady state is necessarily overlapping, as shown in Proposition 4.1. Assume  $\sigma_A > 0$  or  $\sigma_B > 0$ . Suppose that the distributions overlap on Jthat has positive measure. We derive a contradiction.

We first prove  $s_A(V) = \psi s_B(V)$  for all  $V \in J$  and some  $\psi > 0$ , and so the hypotheses in Lemma B.1 are satisfied. Consider any  $V \in J$ . Constant tension in sector -i is equivalent to  $\frac{h'_{-i}}{h_{-i}} = \frac{-s'_{-i}}{s_{-i}}$ . From this equation, subtract (2.16) for sector -i. Using (2.5) to compute  $s'_{-i}$  and (2.6) to compute  $h'_{-i}$ , this subtraction yields

$$\frac{\sigma_{-i}n_iG'_i\left(V\right)}{h_{-i}\left(V\right)} = \frac{\sigma_iF'_i\left(V\right)}{s_{-i}\left(V\right)}.$$

Because  $F'_i(V) > 0$  and  $G'_i(V) > 0$  for all  $V \in J$ , dividing this equation by (2.16) yields

$$\frac{\sigma_{-i}h_{i}\left(V\right)}{h_{-i}\left(V\right)} = \frac{\sigma_{i}s_{i}\left(V\right)}{s_{-i}\left(V\right)}.$$

Substituting  $h_i = \tau_i / s_i$  for i = A, B into the above result yields

$$\sigma_A \tau_B \left[ s_A \left( V \right) \right]^2 = \sigma_B \tau_A \left[ s_B \left( V \right) \right]^2 \text{ for all } V \in J.$$
(B.4)

When  $\sigma_A > 0$  or  $\sigma_B > 0$ , (B.4) can hold only if  $\sigma_A > 0$  and  $\sigma_B > 0$ . For this reason, we assume  $\sigma_A > 0$  and  $\sigma_B > 0$  in the remainder of this proof. Then,  $s_A(V) = \psi s_B(V)$  for all  $V \in J$ , where  $\psi = \left(\frac{\sigma_B \tau_A}{\sigma_A \tau_B}\right)^{1/2}$ . Lemma B.1 implies  $\psi = 1$  and results (i)-(iv). Note that  $\psi = 1$  can be written as  $\tau_A/\tau_B = \sigma_A/\sigma_B$ .

To complete the proof, we consider three cases in turn: Case 1:  $\lambda_A \lambda_B = \sigma_A \sigma_B$ . In this case,  $\lambda_A / \sigma_B = \sigma_A / \lambda_B$ , which is temporarily denoted *a*. Then, (2.5) yields the following equation for all  $V \in J$ :

$$s_A(V) - s_B(V) = (\delta_A - \delta_B) + (\lambda_A - \sigma_A) \left\{ 1 - F_A(V) + \frac{1}{a} \left[ 1 - F_B(V) \right] \right\}.$$

Differentiating this equation and using  $s'_{A}(V) = s'_{B}(V)$ , we get

$$\lambda_A = \sigma_A$$
, and so  $\lambda_B = \sigma_B$ . (B.5)

Under (B.5), the requirement  $s_A(V) = s_B(V)$  for all  $V \in J$  is satisfied only if  $\delta_A = \delta_B$ . With (B.5) and  $\delta_A = \delta_B$ , (B.1) yields the contradiction  $y_A = y_B$ .

For Cases 2 and 3 below,  $\lambda_A \lambda_B \neq \sigma_A \sigma_B$ . In these cases, differentiate (2.5) to solve

$$\begin{pmatrix} F'_{A}(V) \\ F'_{B}(V) \end{pmatrix} = \frac{-s'(V)}{\lambda_{A}\lambda_{B} - \sigma_{A}\sigma_{B}} \begin{pmatrix} \lambda_{B} - \sigma_{B} \\ \lambda_{A} - \sigma_{A} \end{pmatrix} \text{ for all } V \in \text{ interior of } J, \tag{B.6}$$

where we suppressed the subscript of s because  $s_i(V)$  is independent of i for all  $V \in J$ . Since  $\sigma_A > 0$  and  $\sigma_B > 0$ , then  $-s'_i(V) > 0$  for all V in the interior of the support of sector i's distribution (see (A.4)). For  $F_A$  and  $F_B$  to overlap on the set J of positive measure, it is necessary that  $F'_A(V) > 0$  and  $F'_B(V) > 0$  for all V in the interior of J. With (B.6), this requirement is satisfied only if

$$\frac{\lambda_A - \sigma_A}{\lambda_A \lambda_B - \sigma_A \sigma_B} > 0 \text{ and } \frac{\lambda_B - \sigma_B}{\lambda_A \lambda_B - \sigma_A \sigma_B} > 0.$$
(B.7)

Note that (A.4) implies  $-s'_i(\underline{V}_i^+) > 0$  and  $-s'_i(\overline{V}_i^-) > 0$  for i = A, B. If  $\lambda_A \lambda_B \neq \sigma_A \sigma_B$ , (B.6) and (B.7) imply  $F'_A(\underline{V}_A^+) > 0$ ,  $F'_A(\overline{V}_m^-) > 0$  and  $F'_B(\overline{V}_m^-) > 0$ , where  $\overline{V}_m = \min\{\overline{V}_A, \overline{V}_B\}$ . Moreover, using (2.6) to compute  $n_i G'_i$  and substituting  $h_i = \tau_i/s_i$ , we can show that (2.15) is consistent with (B.6) for all  $V \in J$  if and only if  $\sigma_A = \sigma_B$ .

Case 2:  $\lambda_A \lambda_B < \sigma_A \sigma_B$ . We prove  $\bar{V}_A = \bar{V}_B$  in this case. Suppose, to the contrary, that  $\bar{V}_A \neq \bar{V}_B$ . Then,  $\bar{V}_i$  must lie in the interior of  $supp_{-i}$  for either i = A or i = B. Since  $F'_i(\bar{V}_i^-) > 0$ , as shown above, then  $X_{i,-i}(\bar{V}_i) X_{-i,i}(\bar{V}_i) \leq 1$  by Lemma 2.2, where  $X_{i,j}(V)$  is defined in (A.2). The above proof has shown that  $s_i(V) = s_{-i}(V)$  and  $h_i(V)/h_{-i}(V) = \tau_i/\tau_{-i} = \sigma_i/\sigma_{-i}$  for all  $V \in J$ . Using these results, we get

$$X_{-i,i}\left(\bar{V}_{i}\right) = \frac{\sigma_{i}}{\lambda_{-i}}, \quad X_{i,-i}\left(\bar{V}_{i}\right) = \frac{\sigma_{-i}}{\lambda_{i}}.$$

The requirement  $X_{i,-i}(\bar{V}_i) X_{-i,i}(\bar{V}_i) \leq 1$  becomes  $\lambda_i \lambda_{-i} \geq \sigma_i \sigma_{-i}$ , which is violated. Thus,  $\bar{V}_A = \bar{V}_B$  must hold in this case.

With  $\bar{V}_A = \bar{V}_B$ , the requirement  $s_A(\bar{V}) = s_B(\bar{V})$  yields  $\delta_A = \delta_B$ . With  $\delta_A = \delta_B$  and  $\bar{V}_A = \bar{V}_B$ , (B.1) in Lemma B.1 implies the contradiction  $y_A = y_B$ .

Case 3:  $\lambda_A \lambda_B > \sigma_A \sigma_B$ . In this case, (B.7) requires  $\lambda_A > \sigma_A$  and  $\lambda_B > \sigma_B$ . We prove  $\underline{V}_A = \underline{V}_B$  in this case. Suppose  $\underline{V}_A \neq \underline{V}_B$ , to the contrary. Since  $\underline{V}_A \geq \underline{V}_B$  by assumption, then  $\underline{V}_A > \underline{V}_B$ . Since  $F'_i(\underline{V}_A^+) > 0$ , as shown above, then  $F'_B(\underline{V}_A^-) > F'_B(\underline{V}_A^+)$  and  $X_{B,A}(\underline{V}_A) X_{A,B}(\underline{V}_A) \geq 1$  by Lemma 2.2. Because  $s_A(\underline{V}_A) = s_B(\underline{V}_A)$  and  $h_A(\underline{V}_A)/h_B(\underline{V}_A) = \tau_A/\tau_B = \sigma_A/\sigma_B$ , we can compute

$$X_{B,A}(\underline{V}_A) = \frac{\sigma_A}{\lambda_B}, \quad X_{A,B}(\underline{V}_A) = \frac{\sigma_B}{\lambda_A}.$$

The requirement  $X_{B,A}(\underline{V}_A) X_{A,B}(\underline{V}_A) \ge 1$  becomes  $\lambda_A \lambda_B \le \sigma_A \sigma_B$ , which is violated in the current case. Thus,  $\underline{V}_A = \underline{V}_B$  must hold in this case.

With  $\underline{V}_A = \underline{V}_B$ , the requirement  $s_A(\underline{V}) = s_B(\underline{V})$  yields  $\delta_A - \delta_B = \lambda_B - \lambda_A$ , where we have used the result  $\sigma_A = \sigma_B$ . Thus, the steady state is non-overlapping except possibly when  $\lambda_A > \sigma_A$ ,  $\lambda_B > \sigma_B$ ,  $\sigma_A = \sigma_B$  and  $\delta_A - \delta_B = \lambda_B - \lambda_A$ . **QED** 

## C. Proofs for Section 4

We first prove Proposition 4.1. Following the same procedure as the one in the proof of Corollary 3.2, we can verify that  $h_i(V) s_i(V) = \tau_i$  for all  $V \in supp_i$ , where  $\tau_i = \alpha_i u (\delta_i + \lambda_i)$ . Thus, Lemma 3.1 is valid here and  $s_i(V)$  is given by (3.2). To prove  $\underline{V}_A = \underline{V}_B$ , suppose  $\underline{V}_A > \underline{V}_B$ , to the contrary. There exists  $\varepsilon > 0$  such that  $\underline{V}_A - \varepsilon > \underline{V}_B$ . Consider  $V \in (\underline{V}_A - \varepsilon, \underline{V}_A)$ . Then,  $G_A(V) = F_A(V) = 0$ . Because this is true for all V in the interval  $(\underline{V}_A - \varepsilon, \underline{V}_A)$ , then  $G'_A(V) = F'_A(V) = 0$  in this interval, and so  $h'_A(V) = s'_A(V) = 0$ . By the derivation of (2.9) from (2.8), for such V we have

$$\hat{\pi}_{A}^{\prime}\left(V\right)\sim\frac{-h_{A}\left(V\right)}{\hat{\pi}_{A}\left(V\right)m_{A}}<0,$$

where ~ means having the same sign. This result contradicts the optimality condition (2.9) for the case  $V = \underline{V}_A - \varepsilon$ ; that is, reducing the offer from  $\underline{V}_A$  to a slightly lower value strictly increases expected profit of a vacancy. Thus,  $\underline{V}_A = \underline{V}_B$  must hold.

In the limit  $r \to 0$ , (3.2) becomes (3.3). With  $\sigma_A = \sigma_B = 0$ , (3.3) implies the formula of  $F_i$  in (4.2). Since  $\alpha_i u + \lambda_i n_i G_i = h_i = \tau_i / s_i$ , we can solve  $G_i$ . After substituting  $n_i = \frac{\alpha_i u}{\lambda_i \theta_i}$ , which is proven below, the solution of  $G_i$  becomes the formula in (4.2).

In the limit  $r \to 0$ ,  $s_i(V)$  is given by (3.3). We characterize the steady state in Steps (a)-(e) below and prove the remainder of Proposition 4.1. In particular, Steps (d) and (e) establish the necessary and sufficient conditions in (4.1) for the steady state to exist.

**Step (a)**: Solve  $(n_A, n_B, u)$ . Setting  $V = \overline{V}_i$  in the result  $h_i(V) s_i(V) = \tau_i = \alpha_i u (\delta_i + \lambda_i)$ and using  $u = 1 - n_A - n_B$ , we solve the composition of workers in the two sectors and unemployment as in (4.3).

**Step (b)**: Solve  $(\bar{V}_i, \underline{w}_i, \bar{w}_i)$  for  $i \in \{A, B\}$ . Setting  $V = \bar{V}_i$  in (3.3) yields

$$\bar{V}_i - \underline{V}_i = \frac{2m_i\pi_i}{\alpha_i u \left(\theta_i + 1\right)}.$$
(C.1)

Because  $\hat{\pi}_i(V) = \pi_i$  for all  $V \in supp_i$ , then (2.7) and  $h_i = \tau_i/s_i$  imply the following wage function in the limit  $r \to 0$ :

$$w_i(V) = y_i - \frac{m_i \pi_i s_i^2(V)}{\alpha_i u \lambda_i (\theta_i + 1)}, \text{ for all } V \in supp_i.$$
(C.2)

Setting  $V = \underline{V}_i$  and  $V = \overline{V}_i$  in turn, this wage function yields:

$$\underline{w}_{i} = y_{i} - \frac{m_{i}\pi_{i}}{\alpha_{i}u}\lambda_{i}\left(\theta_{i}+1\right), \quad \bar{w}_{i} = y_{i} - \frac{m_{i}\pi_{i}\lambda_{i}\theta_{i}^{2}}{\alpha_{i}u\left(\theta_{i}+1\right)}.$$
(C.3)

Step (c): Solve  $(\underline{V}_i, V_u)$ . Even in the limit  $r \to 0$ , the value  $r\underline{V}_i$  is strictly positive. Setting  $V = \underline{V}_A$  in (2.2) and  $V = \underline{V}_B$  in (2.3), and noticing  $\sigma_A = \sigma_B = 0$ , we get

$$r\underline{V}_{i} = \underline{w}_{i} + \lambda_{i} \int_{\underline{V}_{i}}^{\overline{V}_{i}} \left[1 - F_{i}\left(V\right)\right] dV,$$

where we have used the result  $\underline{V}_i = V_u$  on the right-hand side. To compute the above integral, note that (3.3) implies  $dV = -\frac{2m_i\pi_i}{\tau_i}ds_i$  (V), and the definition of  $s_i$  implies  $1 - F_i = \frac{s_i - \delta_i}{\lambda_i}$ . Substituting these results, we get

$$\int_{\underline{V}_{i}}^{\overline{V}_{i}} \left[1 - F_{i}\left(V\right)\right] dV = -\frac{2m_{i}\pi_{i}}{\lambda_{i}\tau_{i}} \int_{\delta_{i}+\lambda_{i}}^{\delta_{i}} \left(s - \delta_{i}\right) ds,$$

where the integration bounds in the second integral are the value of  $s_i$  at the boundaries of  $supp_i$ . Computing the integral and substituting  $\tau_i = \alpha_i u \left(\delta_i + \lambda_i\right)$ , we obtain

$$\int_{\underline{V}_{i}}^{\overline{V}_{i}} \left[1 - F_{i}\left(V\right)\right] dV = \frac{m_{i}\pi_{i}}{\alpha_{i}u\left(\theta_{i}+1\right)}$$

Substituting this result and  $\underline{w}_i$  from Step (b), the formula of  $r\underline{V}_i$  gives

$$r\underline{V}_i = y_i - \frac{m_i \pi_i \lambda_i \theta_i \left(\theta_i + 2\right)}{\alpha_i u \left(\theta_i + 1\right)}.$$
(C.4)

Similarly, (2.1) yields:

$$rV_u = b + \frac{m_A \pi_A}{u(\theta_A + 1)} + \frac{m_B \pi_B}{u(\theta_B + 1)}.$$
 (C.5)

**Step (d)**: Solve  $(m_A, m_B, \pi)$  and find the conditions for  $m_A, m_B \in (0, 1)$  and  $\pi > 0$ . Subtracting (C.5) from (C.4) and invoking  $\underline{V}_i = V_u$ , we arrive at

$$(y_i - b) u = \frac{m_i \pi_i \lambda_i \theta_i (\theta_i + 2)}{\alpha_i (\theta_i + 1)} + \frac{m_A \pi_A}{\theta_A + 1} + \frac{m_B \pi_B}{\theta_B + 1}.$$

Setting i = A and i = B, we solve the two linear equations:

$$m_{i}\pi_{i} = \frac{u}{\phi}(\theta_{i}+1)\left\{ \left[ \frac{\lambda_{-i}}{\alpha_{-i}}\theta_{-i}(\theta_{-i}+2) + 1 \right](y_{i}-b) - (y_{-i}-b) \right\}$$
(C.6)

where

$$\phi \equiv \frac{\lambda_A}{\alpha_A} \theta_A \left( \theta_A + 2 \right) \left[ \frac{\lambda_B}{\alpha_B} \theta_B \left( \theta_B + 2 \right) + 1 \right] + \frac{\lambda_B}{\alpha_B} \theta_B \left( \theta_B + 2 \right).$$

Add up (C.6) for i = A and i = B. Since  $m_A + m_B = 1$  and  $\pi_A = \pi_B = \pi$ , we get  $\pi$ . Substituting the solution of  $\pi$  for  $\pi_i$  in (C.6), we obtain  $m_i$ .

For  $m_A, m_B \in (0, 1)$  and  $\pi > 0$ , it is necessary that  $m_i \pi_i > 0$  for both i = A and i = B. Using (C.6), we can write these necessary requirements as (4.1). Also, (4.1) is sufficient for  $\pi > 0$  and  $m_i \in (0, 1)$  for i = A, B. To see why, suppose that (4.1) holds, so that  $m_A \pi_A > 0$  and  $m_B \pi_B > 0$ . Then,  $\pi = m_A \pi_A + m_B \pi_B > 0$ . Since  $m_A \pi > 0$ ,  $m_B \pi > 0$  and  $\pi > 0$ , then  $m_A > 0$  and  $m_B > 0$ . Because  $m_A + m_B = 1$ , then  $m_A < 1$  and  $m_B < 1$ . Thus, (4.1) is necessary and sufficient for  $m_A > 0, m_B > 0$  and  $\pi > 0$ .

Step (e): Find the conditions under which deviations outside the supports of the distributions are not profitable. The procedure of analyzing such deviations is similar to that in the proof of Lemma 2.2 in Appendix A. However, since  $\sigma_A = \sigma_B = 0$  and  $\underline{V}_A = \underline{V}_B$  in the current case, the procedure yields no additional restrictions on the steady state. Thus, either  $\bar{V}_A \geq \bar{V}_B$  or  $\bar{V}_A < \bar{V}_B$  is possible. This establishes Proposition 4.1.

Now we prove Proposition 4.2. Substituting  $m_i \pi_i$  from (C.6) into (C.3) and comparing the outcome for i = A with the outcome for i = B, we can verify (i) and (ii). To prove (iii), use  $h_i = \tau_i/s_i$  to derive  $dG_i(V) = -\frac{\theta_i \tau_i}{\alpha_i u} s_i^{-2} ds_i$ . Substituting this result and (C.2) into the definition of  $\mathbb{E}w_i$ , we integrate to obtain

$$\mathbb{E}w_i = y_i - \frac{m_i \pi_i \lambda_i \theta_i}{\alpha_i u}.$$
(C.7)

With this formula and (C.6), it is straightforward to verify (iii). For (iv),  $F_A$  first-order stochastically dominates  $F_B$  if and only if  $F_A(V) \leq F_B(V)$  for all V, where the inequality is strict for a positive measure of values on the support. With the formula of  $F_i$  in (4.2), this stochastic dominance is equivalent to  $\frac{m_A \pi_A}{\alpha_A(\theta_A+1)} > \frac{m_B \pi_B}{\alpha_B(\theta_B+1)}$  which, by (C.1), is equivalent to  $\bar{V}_A > \bar{V}_B$ . Substituting  $m_i \pi_i$  from (C.6) into (C.1) and comparing  $\bar{V}_A$  with  $\bar{V}_B$ , we can verify that  $\bar{V}_A > \bar{V}_B$  if and only if (4.6) holds. Moreover, with the formula of  $G_i$  in (4.2), it is clear that  $G_A$  first-order stochastically dominates  $G_B$  in the case  $\theta_A = \theta_B$  if and only if  $\frac{m_A \pi_A}{\alpha_A} > \frac{m_B \pi_B}{\alpha_B}$ , which is equivalent to (4.6) in this case.

Corollary 4.3 can be verified directly with the above expressions for  $(\bar{w}_i, \underline{w}_i, \mathbb{E}w_i, \bar{V}_i)$ . QED

## D. Proof of Proposition 5.1

Assume that the distributions in the two sectors do not overlap. Then,  $\underline{V}_A = \overline{V}_B$  by Lemma 2.2. Setting i = A in (2.5) and (2.6) and considering any  $V \in supp_A$ , we get

$$s_{A}(V) = \delta_{A} + \lambda_{A} [1 - F_{A}(V)],$$
  

$$h_{A}(V) = \alpha_{A}u + \sigma_{A}n_{B} + \lambda_{A}n_{A}G_{A}(V)$$

Differentiation yields  $F'_A(V) = -s'_A(V)/\lambda_A$  and  $n_A G'_A(V) = h'_A(V)/\lambda_A$ . Substituting these results into (2.15) yields  $[h_A(V) s_A(V)]' = 0$ . Thus,  $h_A(V) s_A(V) = \tau_A$  for all  $V \in supp_A$ , where  $\tau_A = h_A(\underline{V}_A) s_A(\underline{V}_A)$ . Similarly,

$$s_B(V) = \delta_B + \sigma_A + \lambda_B [1 - F_B(V)]$$
  
$$h_B(V) = \alpha_B u + \lambda_B n_B G_B(V).$$

We can deduce  $h_B(V) s_B(V) = \tau_B$  for all  $V \in supp_B$ , where  $\tau_B = h_B(\underline{V}_B) s_B(\underline{V}_B)$ .

To find the conditions for the non-overlapping steady state to exist, we characterize the steady state in the limit  $r \to 0$ . Steps (a)-(d) are similar to those in the proof of Proposition 4.1, with the modification  $\underline{V}_A = \overline{V}_B$  instead of  $\underline{V}_A = \underline{V}_B$ , but Step (e) is different. Step (a) yields  $(n_A, n_B, u)$  and

$$\tau_A = u \left(\delta_A + \lambda_A\right) \left(\alpha_A + \frac{\sigma_A \alpha_B}{\delta_B + \sigma_A}\right), \ \tau_B = u \alpha_B \left(\delta_B + \sigma_A + \lambda_B\right). \tag{D.1}$$

Step (b) yields

$$\bar{V}_{i} - \underline{V}_{i} = \frac{m_{i}\pi_{i}}{\tau_{i}}\lambda_{i}.$$
(D.2)
$$\underline{w}_{A} = y_{A} - (\delta_{A} + \lambda_{A})^{2} \frac{m_{A}\pi_{A}}{\xi_{A}}, \quad \bar{w}_{A} = y_{A} - \delta_{A}^{2} \frac{m_{A}\pi_{A}}{\xi_{A}},$$

$$\underline{w}_{B} = y_{B} - (\delta_{B} + \sigma_{A} + \lambda_{B})^{2} \frac{m_{B}\pi_{B}}{\xi_{B}}, \quad \bar{w}_{A} = y_{B} - (\delta_{B} + \sigma_{A})^{2} \frac{m_{B}\pi_{B}}{\xi_{B}}.$$
(D.2)

Note that  $\bar{V}_i - \underline{V}_i \in (0, \infty)$  even though  $\bar{V}_i$  and  $V_i \to \infty$  when  $r \to 0$ . Step (c) yields

$$r\underline{V}_{A} = y_{A} - \delta_{A} \left(\delta_{A} + 2\lambda_{A}\right) \frac{m_{A}\pi_{A}}{\tau_{A}} - 2\lambda_{B}\delta_{A} \frac{m_{B}\pi_{B}}{\tau_{B}}$$

$$r\bar{V}_{B} \rightarrow r\underline{V}_{B} = y_{B} + \lambda_{A}\sigma_{A} \frac{m_{A}\pi_{A}}{\tau_{A}} - \left[\left(\delta_{B} + \sigma_{A}\right)^{2} + 2\lambda_{B}\delta_{B}\right] \frac{m_{B}\pi_{B}}{\tau_{B}}$$

$$rV_{u} = b + \lambda_{A}\alpha_{A} \frac{m_{A}\pi_{A}}{\tau_{A}} + \lambda_{B} \left(\alpha_{B} + 2\alpha_{A}\right) \frac{m_{B}\pi_{B}}{\tau_{B}},$$

$$(D.4)$$

where we have substituted  $(\bar{V}_B - \underline{V}_B)$  from (D.2) into the right-hand side of the Bellman equations. Note that  $r\bar{V}_B \rightarrow r\underline{V}_B$  because  $r \rightarrow 0$ , although  $\bar{V}_B > \underline{V}_B$ . In step (d), we invoke the requirements for the non-overlapping steady state,  $\underline{V}_A = \bar{V}_B$  and  $\underline{V}_B = V_u$ . With (D.4), these requirements solve

$$\begin{pmatrix} m_A \pi_A / \tau_A \\ m_B \pi_B / \tau_B \end{pmatrix} = \frac{1}{\phi} \begin{pmatrix} z_1, & z_2 \\ z_3, & z_4 \end{pmatrix} \begin{pmatrix} y_A - y_B \\ y_B - b \end{pmatrix},$$
 (D.5)

where  $\phi = z_1 z_4 - z_2 z_3$  (different from  $\phi$  in the proof of Proposition 4.1) and

$$z_{1} = (\delta_{B} + \sigma_{A})^{2} + \lambda_{B} (2\delta_{B} + \alpha_{B} + 2\alpha_{A}), \quad z_{2} = (\delta_{B} + \sigma_{A})^{2} + 2\lambda_{B} (\delta_{B} - \delta_{A})$$
  

$$z_{3} = \lambda_{A} (\sigma_{A} - \alpha_{A}), \quad z_{4} = \delta_{A} (\delta_{A} + 2\lambda_{A}) + \lambda_{A} \sigma_{A}.$$
(D.6)

Similar to the proof of Proposition 4.1,  $m_A, m_B \in (0, 1)$  and  $\pi > 0$  if and only if  $m_A \pi_A / \tau_A > 0$  and  $m_B \pi_B / \tau_B > 0$ . It can be verified that  $\phi > 0$ ,  $z_1 > 0$  and  $z_4 > 0$ . Then,  $m_A \pi_A / \tau_A > 0$  and  $m_B \pi_B / \tau_B > 0$  if and only if (5.1) holds, where

$$k_1 = \frac{-z_2}{z_1}, \quad k_2 = \frac{-z_3}{z_4}.$$
 (D.7)

Step (e) yields (5.2) in Proposition 5.1. We find the condition under which it is not profitable for a firm in sector A to deviate below  $\underline{V}_A$  or for a firm in sector B to deviate above  $\overline{V}_B$ . The procedure is similar to that in the proof of Lemma 2.2 in Appendix A, but the results differ because the two distributions do not overlap in the current case. Consider first a firm in sector A that deviates to  $\underline{V}_A^-$ . For this deviation not to be profitable, the required condition  $\hat{\pi}'_A(\underline{V}_A^-) \geq 0$  is the second case of (2.9) for i = A. We calculate the derivatives  $h'_A(\underline{V}_A^-)$  and  $s'_A(\underline{V}_A^-)$  in the condition. Because  $F_A(V) = G_A(V) = 0$  for all  $V < \underline{V}_A$ , then  $F'_A(\underline{V}_A^-) = G'_A(\underline{V}_A^-) = 0$ . (2.5) for i = B implies

$$F'_B\left(\underline{V}_A^-\right) = \frac{-1}{\lambda_B} s'_B\left(\underline{V}_A^-\right).$$

Differentiating (2.5) and (2.6) for i = A, we get

$$s'_{A}\left(\underline{V}_{A}^{-}\right) = -\sigma_{B}F'_{B}\left(\underline{V}_{A}^{-}\right) = \frac{\sigma_{B}}{\lambda_{B}}s'_{B}\left(\underline{V}_{A}^{-}\right),$$
$$h'_{A}\left(\underline{V}_{A}^{-}\right) = \sigma_{A}n_{B}G'_{B}\left(\underline{V}_{A}^{-}\right) = \frac{\sigma_{A}h_{B}\left(\underline{V}_{A}\right)}{s_{B}\left(\underline{V}_{A}\right)}F'_{B}\left(\underline{V}_{A}^{-}\right) = \frac{-\sigma_{A}\tau_{B}}{\lambda_{B}s^{2}_{B}\left(\underline{V}_{A}\right)}s'_{B}\left(\underline{V}_{A}^{-}\right).$$

The second equality in the result for  $h'_A$  uses (2.15) and the third equality uses constant tension. Note that the superscript - on  $\underline{V}_A$  is dropped from h and s because these functions are continuous. Substituting these results, we can write the second case of (2.9) for i = A, i.e.,  $\hat{\pi}'_A(\underline{V}_A) \ge 0$ , as

$$0 \leq \frac{-s_B'\left(\underline{V}_A^-\right)}{\lambda_B} \left[\frac{\tau_B \sigma_A s_A\left(\underline{V}_A\right)}{\tau_A s_B^2\left(\underline{V}_A\right)} + \frac{\sigma_B}{s_A\left(\underline{V}_A\right)}\right] - \frac{h_A\left(\underline{V}_A\right)}{m_A \pi_A},$$

where we have used the limit  $r \to 0$ . For any arbitrarily small  $\varepsilon > 0$ , since  $(\underline{V}_A - \varepsilon)$  lies in the interior of  $supp_B$ , the first-order condition for a firm in sector B holds as equality at this offer. With constant tension, this implies that (A.4) holds at  $\underline{V}_A^-$  for i = B. When  $r \to 0$ , (A.4) implies  $s'_B(\underline{V}_A^-) = \frac{-\tau_B}{2m_B\pi_B}$ . Substituting this result and substituting  $(s_A(\underline{V}_A), s_B(\underline{V}_A))$  from (2.5), we rewrite the above condition for  $\hat{\pi}'_A(\underline{V}_A^-) \ge 0$  as

$$\frac{m_A \pi_A / \tau_A}{m_B \pi_B / \tau_B} \ge k_5 \equiv 2\lambda_B \left[ \frac{\tau_B \sigma_A}{\tau_A} \left( \frac{\delta_A + \lambda_A}{\delta_B + \sigma_A} \right)^2 + \sigma_B \right]^{-1}$$

Similarly, we can consider a firm in sector B that deviates to the offer  $\bar{V}_B^+$ . For this deviation not to be profitable, the required condition  $\hat{\pi}'_B(\bar{V}_B^+) \leq 0$  is the third case of (2.9) for i = B. Adapting the above procedure to this case, we can express the condition as

$$\frac{m_A \pi_A / \tau_A}{m_B \pi_B / \tau_B} \ge k_6 \equiv \frac{1}{2\lambda_A} \left[ \frac{\tau_A \sigma_B}{\tau_B} \left( \frac{\delta_B + \sigma_A}{\delta_A + \lambda_A} \right)^2 + \sigma_A \right].$$

Define  $k_7 = \max\{k_5, k_6\}$  and note  $k_7 > 0$ . Then,  $\hat{\pi}'_A(\underline{V}_A^-) \ge 0$  and  $\hat{\pi}'_B(\overline{V}_B^+) \le 0$  if and only if  $\frac{m_A \pi_A/\tau_A}{m_B \pi_B/\tau_B} \ge k_7$ . Under (D.5), this condition is equivalent to (5.2) where

$$k_3 = z_1 - k_7 z_3, \quad k_4 = z_2 - k_7 z_4.$$
 (D.8)

To prove that  $\alpha_A \geq \sigma_A$  is sufficient for the existence region to be non-empty in the parameter space, suppose  $\alpha_A \geq \sigma_A$ . Then,  $z_3 \leq 0$  and  $k_2 \geq 0$ . Moreover, since  $k_3 > 0$  in

this case, (5.2) can be rewritten as  $\frac{y_A - y_B}{y_B - b} \ge \frac{-k_4}{k_3}$ . Because  $\phi = z_1 z_4 - z_2 z_3 > 0$ , it can be verified that  $\frac{-k_4}{k_3} > k_1$  in this case, and so (5.1) and (5.2) are equivalent to

$$\frac{y_A - y_B}{y_B - b} \ge \max\{\frac{-k_4}{k_3}, \varepsilon\}, \quad \frac{y_B - b}{y_A - y_B} > k_2, \tag{D.9}$$

where  $\varepsilon > 0$  is an arbitrarily small number. If  $k_4 \ge 0$ , the first condition in (D.9) is satisfied by all  $y_A > y_B$  and the second condition is satisfied in an non-empty interval of  $(y_A, y_B)$ . If  $k_4 < 0$ , (D.9) is equivalent to

$$\frac{k_3}{-k_4} \ge \frac{y_B - b}{y_A - y_B} > k_2.$$

Since  $z_1 z_4 > z_2 z_3$ , then  $\frac{k_3}{-k_4} > k_2$ , and so the above interval for  $\frac{y_B - b}{y_A - y_B}$  is non-empty.

Using (D.3), we can derive the condition for  $\underline{w}_A < \overline{w}_B$  as

$$0 > y_A - y_B - (\delta_A + \lambda_A)^2 \frac{m_A \pi_A}{\tau_A} + (\delta_B + \sigma_A)^2 \frac{m_B \pi_B}{\tau_B}$$

In the case  $\delta_A = \delta_B$ , substituting (D.5) into the above condition yields

$$0 > (\sigma_A - \lambda_A) \left[ (\delta_B + \sigma_A)^2 + \lambda_B (2\delta_B + \alpha_B + 2\alpha_A) \right] (y_A - y_B) + (\sigma_A - \lambda_A) (\delta_B + \lambda_A)^2 (y_B - b).$$

Thus,  $\delta_A = \delta_B$  and  $\sigma_A < \lambda_A$  are sufficient for  $\underline{w}_A < \overline{w}_B$ .

To find the effect of an increase in b on the spreads in wage rates and values, note from (D.2) and (D.3) that  $\frac{d}{db}(\bar{w}_A - \underline{w}_A) < 0$  and  $\frac{d}{db}(\bar{V}_A - \underline{V}_A) < 0$  if and only if  $\frac{d}{db}\left(\frac{m_A\pi_A}{\tau_A}\right) < 0$ . With (D.5), this condition is equivalent to  $z_2 > 0$ , which can be written as  $2\lambda_B (\delta_A - \delta_B) < (\delta_B + \sigma_A)^2$ . Similarly,  $\frac{d}{db}(\bar{w}_B - \underline{w}_B) < 0$  and  $\frac{d}{db}(\bar{V}_B - \underline{V}_B) < 0$  if and only if  $\frac{d}{db}\left(\frac{m_B\pi_B}{\tau_B}\right) < 0$ , which is equivalent to  $z_4 > 0$  and is always satisfied. The ratios  $(\bar{w}_A - \underline{w}_A) / (\bar{w}_B - \underline{w}_B)$  and  $(\bar{V}_A - \underline{V}_A) / (\bar{V}_B - \underline{V}_B)$  increase in b if and only if  $\frac{d}{db}\left(\frac{m_A\pi_A/\tau_A}{m_B\pi_B/\tau_B}\right) > 0$ . With (D.5), the latter condition is equivalent to  $z_1z_4 - z_2z_3 > 0$ , which is satisfied. **QED** 

# References

- Beaudry, P., Green, D., Sand, B., 2012. Does Industrial Composition Matter for Wages? A Test of Search and Bargaining Theory, Econometrica 80, 1063-1104.
- [2] Bontemps, C., Robin, J.M., van den Berg, G.J., 2000. Equilibrium Search with Continuous Productivity Dispersion: Theory and Nonparametric Estimation, International Economic Review 41, 305-358.
- [3] Bradley, J., Postel-Vinay, F., Turon, H., 2014. Public Sector Wage Policy and Labor Market Equilibrium: A Structural Model. Manuscript, University College London.
- [4] Burdett, K., 2012. Towards a Theory of Labor Market Equilibrium with a Public Sector, Labour Economics 19, 68-75.
- [5] Burdett, K., Coles, M.G., 2003. Equilibrium Wage-Tenure Contracts, Econometrica 71, 1377-1404.
- [6] Burdett, K., Judd, K., 1983. Equilibrium Price Dispersion, Econometrica 51, 955-970.
- [7] Burdett, K., Mortensen, D., 1998. Wage Differentials, Employer Size, and Unemployment, International Economic Review 39, 257-273.
- [8] Delacroix, A., Shi, S., 2006. Directed Search On the Job and the Wage Ladder, International Economic Review 47, 651-699.
- [9] Hoffmann, F., Shi, S., 2011. Sectoral Shift, Job Mobility and Wage Inequality. Manuscript, University of Toronto.
- [10] Jolivet, G., Postel-Vinay, F., Robin, J.M., 2006. The Empirical Content of the Job Search Model: Labor Mobility and Wage Distributions in Europe and the US, European Economic Review 50, 877-907.
- [11] Lee, D., Wolpin, K., 2006. Intersectoral Labor Mobility and the Growth of the Service Sector, Econometrica 74, 1-46.
- [12] Lise, J., Robin, J.M., 2013. The Macro-Dynamics of Sorting between Workers and Firms. Manuscript, University College London.
- [13] Meghir, C., Narita, R., Robin, J.M., 2015. Wages and Informality in Developing Countries, American Economic Review 105, 1509-1546.

- [14] Menzio, G., Shi, S., 2011. Efficient Search on the Job and the Business Cycle, Journal of Political Economy 119, 468-510.
- [15] Mortensen, D., 2005. Wage Dispersion: Why Are Similar Workers Paid Differently? MIT Press, Cambridge, M.A.
- [16] Moscarini, G., Postel-Vinay, F., 2013. Did the Job Ladder Fail After the Great Recession? Manuscript, Yale University.
- [17] Postel-Vinay, F., Robin, J.M., 2002. Equilibrium Wage Dispersion with Worker and Employer Heterogeneity, Econometrica 70, 2295-2350.
- [18] Shi, S., 2009. Directed Search for Equilibrium Wage-Tenure Contracts, Econometrica 77, 561-584.
- [19] van den Berg, G.J., Ridder, G., 1998. An Equilibrium Search Model of the Labor Market, Econometrica 66, 1183-1221.